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Local Rigidity and Self-Organized Criticality for Avalanches.

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Abstract. – The general conditions for a sandpile system to evolve spontaneously into a critical state characterized by a power law distribution of avalanches or bursts are identified as: *a*) the existence of a stationary state with a global conservation law; *b*) long-range correlations in the continuum limit (*i.e.* Laplacian diffusive field); *c*) the existence of a local rigidity for the microscopic dynamics. These conditions permit a classification of the models that have been considered up to now and the identification of the local rigidity as a new basic parameter that can lead to various possible scenarios ranging continuously from SOC behaviour to standard diffusion.

Introduction. – The term Self-Organized Criticality (SOC) refers to systems which evolve spontaneously into a critical state [1]. This situation is different from usual critical phenomena in which a fine tuning of a critical parameter is necessary to reach the critical point. A fundamental difference between these two classes of phenomena is the role of the dynamical evolution that in usual critical phenomena is eliminated in view of ergodicity. In SOC systems, instead, one deals with open non-linear systems in which the temporal evolution has to be explicitly considered. For these reasons it is believed that SOC systems require the development of theoretical concepts of novel type.

As for *the nature of the critical state* in a SOC system we face essentially two broad classes. In fractal growth phenomena [2] the critical state consists in the fact that the final structure is fractal with scale-invariant correlations. In sandpile models the critical state refers instead to the fact that a disturbance can lead to avalanches of any size characterized by a power law distribution. For these last models spatial correlations are usually short ranged and the avalanches are compact (non-fractal), at least in 2d [3].

The crucial point is to define a theoretical framework in which one should be able to classify and describe all these phenomena. Various questions concerning both fractal growth and sandpile models can be addressed within the new approach of the Fixed-Scale Transformation (FST) [4, 5].

In this letter we intend to address a different and more general question in relation to the physics of avalanche phenomena. In particular, we try to define the general conditions under which sandpile systems evolve spontaneously into a state that leads to a critical distribution of avalanches.

Our specific discussion will be based on the usual sandpile [1, 6] models but we are going to see that the conclusions will be more general. In this model a real variable $E(x, y)$ that we call energy is assigned to each site of a square lattice. If at a certain time t a site exceeds the critical value E_c ($E(x, y; t) > E_c$) the site becomes unstable and relaxes and at the next time step the system is updated in the following way:

$$\begin{cases} E(x, y; t+1) = 0, \\ E(x, y \pm 1; t+1) = E(x, y \pm 1; t) + E(x, y; t)/4, \\ E(x \pm 1, y; t+1) = E(x \pm 1, y; t) + E(x, y; t)/4. \end{cases} \tag{1}$$

In this way a relaxation event can trigger other relaxations and an avalanche can be generated. After the complete relaxation of an avalanche some energy δE is added to a randomly selected site. The relevant quantity for the system is $E_c/\delta E$. In our simulations we keep δE fixed ($\delta E = 1/4$) and vary E_c.

The dynamical equation for this type of model is

$$E(i; t+1) - E(i; t) = -\theta[E(i; t) - E_c]E(i; t) + \sum_{\mathrm{NN}} \frac{\theta[E(j; t) - E_c]E(j; t)}{2d} + \eta(i; t), \tag{2}$$

where $E(i; t)$ is the energy contained in the i-th cell, d is the space dimension, E_c is a critical energy that characterizes, for δE fixed, the microscopic rigidity of the system, η is the noise and θ is the Heaviside step function. This dynamics is usually referred to an open system in which, on average, the energy injected equals the one released at the boundaries. This defines a stationarity condition.

The continuum limit of this dynamic equation poses very interesting questions. The parameter space within which one can define a continuum limit is (E_c, l) where l, the minimal scale for the dynamics, is the size of the cells one considers and in eqs. (1) we have $l = 1$.

The continuum limit in the parameter space (E_c, l) can be taken in three different ways. Let us first consider $E_c \to 0$ followed by $l \to 0$. In this case we have that $\theta[E(i; t)] = 1$ for any value of E and the right-hand side of eq. (2) becomes the discretized Laplacian. Then, by making the limit $l \to 0$ (and going to continuous time considering the time step proportional to l^d) we obtain

$$\frac{\partial E(\mathbf{r}, t)}{\partial t} = \frac{1}{2d} \nabla^2 (E(\mathbf{r}, t)) + \eta(\mathbf{r}, t). \tag{3}$$

The problem reduces therefore to a diffusive field that is perturbed by the noise term η. Note that one reaches the same result by exchanging the order of the two limits.

Despite the familiarity of eq. (3) it is interesting to define the stationarity condition and the concept of avalanches in such a case. Dropping slowly water in a container full to the brim, each drop falling down will produce a perturbation on the surface of the water that will propagate according to eq. (3). Step by step the conservation is assured because for each drop incoming there will be a drop coming out of the container, no matter how large this is. This is because the Laplace operator induces long-range correlations and, in this perspective, all avalanches are infinite. Note that in these problems the propagation of a single avalanche is considered as a fast event while a second time scale governs the global behaviour of many

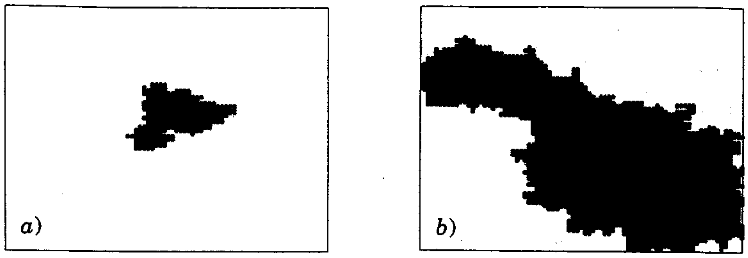

Fig. 1. – *a*) Localized avalanche; *b*) large avalanche carrying energy out of the system.

avalanches. In summary the continuum limit of eq. (3) corresponds, in the framework of SOC problems, to a distribution of avalanches defined by a delta-function peaked at infinity or at the maximum avalanche size given by L^d, where L is the system size. This discussion clarifies that long-range correlations in these problems are sort of trivial and depend on the reminiscence of the Laplacian behaviour. This perspective is different from the usual one, inspired by the properties of phase transitions, according to which it is difficult to see why a system with short-range couplings as eq. (2) leads finally to long-range correlations. In fact the Landau-Ginzburg continuum evolution of a magnetic system is [7]

$$\frac{\partial M(r, t)}{\partial t} = -\nabla^2 M(r, t) + \lambda M(r, t) - \gamma M^3(r, t) + \ldots , \qquad (4)$$

where M is the magnetization. For generic values of the parameters λ and γ, correlations decay exponentially because a screening or «mass» term is present in the problem. Only at $T = T_c$ the linear and cubic term compensate exactly, the system becomes critical and correlations show power law behaviour.

In SOC problems there are no screening effects to the Laplacian field but the important element is, as we are going to see, the local rigidity, which in our case (δE fixed) corresponds to the threshold E_c in eq. (2). In general one could define the local rigidity as $E_c/\delta E$.

From our perspective, therefore, the point is not how to generate long-range correlations from short-range couplings, but rather to understand how the infinite large avalanches, corresponding to the continuum limit (3), are turned into a power law distribution over all scales for eq. (2). Due to the fact that the local rigidity introduces a characteristic length in the system, corresponding to the cell size l, in principle one could expect that the distribution of avalanches should be localized around this length. However, in the stationary state this is impossible because, in a situation in which disturbances lead to various small-scale avalanches (fig. 1*a*)), there will be local accumulation of energy. After further addition of energy at the same point, it is unavoidable that a large avalanche occurs (fig. 1*b*)) in order to dissipate the energy outside the system and ensure the stationarity of the process. Therefore the distribution of avalanche size will necessarily extend from l to infinity.

In order to test these concepts we have performed computer simulations in which the microscopic rigidity can be tuned gradually from a situation corresponding to eq. (2) to one corresponding to eq. (3). This can be achieved, for the dynamical system described by eqs. (1), by varying the value of E_c (keeping δE fixed) from about zero to one. In fig. 2 we show the results for the avalanche size distribution $P(s)$, where s is the total number of sites involved in an avalanche. For $E_c = 1$ we have the standard power law behaviour of SOC avalanches. By decreasing the value of E_c the avalanche distribution changes its nature and large avalanches become dominant with respect to small ones. This tendency is systematic until for very small values of E_c ($E_c = 0.001$) the maximum of the distribution corresponds to

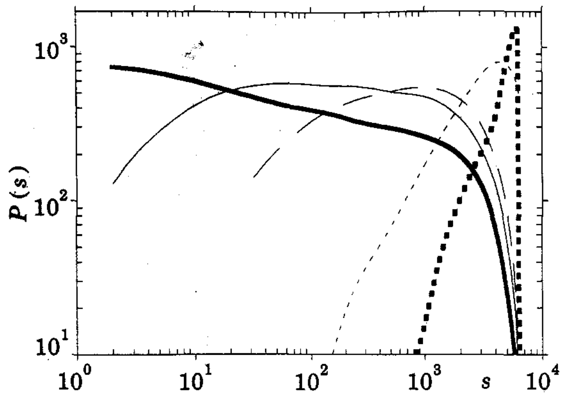

Fig. 2. – Avalanche size distribution for different values of the microscopic rigidity E_c. For $E_c = 1$ (thick solid line) we recover the usual SOC behaviour and the power law distribution of avalanches. By lowering the local rigidity (E_c) (—— $E_c = 0.25$, — — — $E_c = 0.1$, - - - - $E_c = 0.01$) the nature of the avalanche distribution gradually changes until, for $E_c \to 0$ (thick dashed line), one recovers a simple diffusive behaviour with only infinite avalanches. The system size in our simulations is 80×80.

avalanches that are as large as the entire system. This behaviour confirms our analysis on the crucial role of the microscopic rigidity and the fact that infinitely large avalanches are related to the trivial continuum limit.

In order to take a continuum limit ($l \to 0$) which maintains the SOC behaviour in the system, it is necessary to preserve the local rigidity at all scales. Mathematically this implies that the θ-functions must survive to this limit. This can be obtained by considering $l \to 0$ and $E_c \to 0$ in such a way that

$$\lim_{l \to 0} \lim_{E_c \to 0} \frac{E_c}{l^d} = \text{const} \tag{5}$$

and we can consider this as a sort of thermodynamic limit for SOC avalanche problems.

The energy $E(i; t)$ can be viewed as the energy contained in a cell of size l centred around the site i. We can introduce an *energy density function* $\varepsilon(\boldsymbol{r}, t)$ such that $E(i; t) = \int_{i,l} \varepsilon(\boldsymbol{r}, t)\, \mathrm{d}^d r$.

In the limit $l \to 0$ we can make the assumption that $\varepsilon(\boldsymbol{r}, t)$ is constant inside the cell and we can write: $E(i; t) \simeq l^d \varepsilon(\boldsymbol{r}, t)$.

For the same reason we have $E_c \simeq l^d \varepsilon_c$. The noise term, proportional to δE, can be rescaled in a similar way: $\eta(i; t) \simeq l^d \eta_r(\boldsymbol{r}_i, t)$.

By substituting these relations into eq. (2), we obtain after some algebra

$$\varepsilon(\boldsymbol{r}, t+1) - \varepsilon(\boldsymbol{r}, t) = \frac{1}{2d} \nabla^2 \big(\theta(\varepsilon(\boldsymbol{r}, t) - \varepsilon_c)\, \varepsilon(\boldsymbol{r}, t)\big) + \eta_r(\boldsymbol{r}, t), \tag{6}$$

that is the stochastic iteration that, by preserving the local rigidity in the small-scale limit, correctly extends the SOC behaviour of avalanches from $l \to 0$ to infinity. Eventually one can also consider the continuum time limit to obtain a stochastic equation [8], but this leads to a subtle question for the appropriate definition of the noise in this limit.

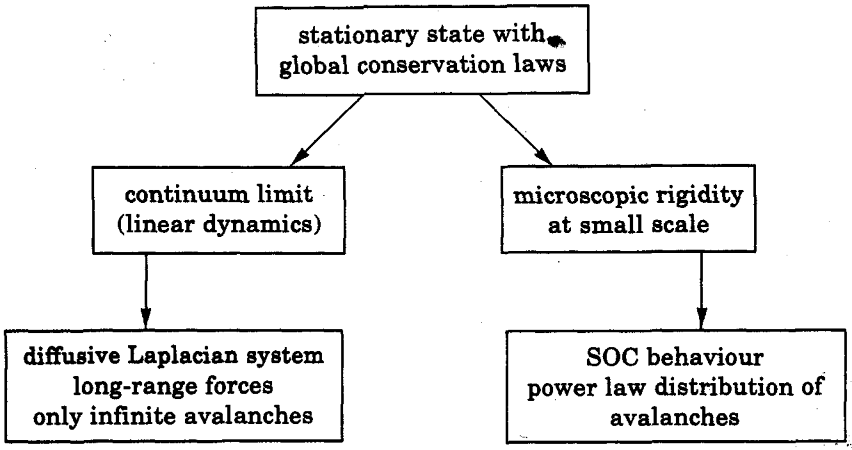

Fig. 3. – Scheme of conditions that illustrate how a local rigidity turns a diffusive system into a SOC system with avalanches of all sizes.

Our scheme of conditions (sketched in fig. 3) refers to a system with a stationary state that satisfies a global conservation law in the sense that what goes in must go out from its boundaries sooner or later. The trivial continuum limit leads to a linear diffusive problem characterized by long-range correlations and by infinite avalanches (in a trivial sense) without fluctuations. The condition of microscopic rigidity at small scale for the dynamics leads instead to a scale-invariant SOC distribution of avalanches that extends from this scale to infinity. The problem is highly non-linear and in this case the avalanches also correspond to the fluctuations of the system whose intensity can be very large. These general conditions refer to the origin of self-organization in the system and the generation of an extended distribution of avalanches. For the detailed calculation of the exponents that characterize the distribution of avalanches and the dynamical evolution it is necessary to consider a specific renormalization scheme of novel type as discussed in ref. [9].

The local rigidity, then, plays a crucial role in identifying various phenomena and in depicting different scenarios. The limit of large local rigidity will lead to the usual SOC behaviour. For lower values one obtains an intermediate behaviour and for very small values one recovers the trivial diffusive behaviour. It is interesting to consider the various SOC models considered up to now from this new point of view:

a) *Earthquake models and acoustic emission*: the case of acoustic emission and earthquakes [10] is directly related to sandpile models (apart from being non-conservative models). In particular earthquakes are produced by an accumulation of stress due to tectonic motion. The static friction between tectonic faults plays the role of the local rigidity, while long-range correlations are induced by the elastic field that redistributes the stress.

b) *Economics*: Self-Organized Criticality has been applied to simple economical models in order to capture the basic elements of the observed instability of economic aggregates [11]. In an economy driven by independent fluctuations in the demand for final goods, exogenous shocks (input), the microscopic rigidity can be identified in the maximum number of orders (stock) that a unit can fulfil without producing and sending orders to other units. In general, an external input will generate an avalanche defined as a sequence of orders between production units.

c) CDW: Charge Density Waves (CDW), pinned by random impurities [12], are an interesting example in which the concept of local rigidity can help to make some predictions. The increase of the external field in usual CDW drives the system but, at the same time, it reduces the rigidity. When the field reaches the threshold the CDW becomes depinned and a single infinite avalanche is present. In order to obtain avalanches of all sizes, it is necessary to preserve the rigidity while driving the system. This can be obtained by pulling the CDW from one end instead of applying a uniform external field. We expect, therefore, that, in this situation, one should observe a SOC behaviour.

d) *Biological models*: simple evolution models like the Bak and Sneppen model [13] or the Game of Life have been proposed in connection to Self-Organized Criticality. It seems that the concept of local rigidity cannot be identified in these systems. Therefore, we conjecture that the mechanism generating avalanches of all sizes in these models should be a different one. The same situation should apply to the problem of Invasion Percolation [14].

e) *Forest fires*: Forest Fires Models (FFM), initially introduced by Bak *et al.* [15] and after modified by Drossel and Schwabl [16], were supposed to show Self-Organized Criticality. These models have more than one field, and therefore they go beyond our simple discussion. However, also in this case, a local rigidity cannot be identified. So we believe they should be qualitatively very different from standard sandpiles. In fact, from our recent studies [17], we discovered the existence, for such models, of a relevant parameter corresponding to a repulsive fixed point for the renormalization group equations. The existence of this parameter places the FFM more in the well-known field of ordinary critical phenomena rather than in the self-organized critical systems.

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