Dynamical properties and predictability of a class of self-organized critical models

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We consider a particular class of self-organized critical models. For these systems we show that the Lyapunov exponent is strictly lower than zero. That allows us to describe the dynamics in terms of a piecewise linear contractive map. We describe the physical mechanisms underlying the approach to the recurrent set in the configuration space and we discuss the structure of the attractor for the dynamics. Finally the problem of the chaoticity of these systems and the definition of a predictability are addressed.

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The wide class of systems in nature which show self-organized criticality (SOC) [1–3] has attracted much attention in the last few years. In particular, sandpile models [4] represent an interesting example of a SOC system; that is, of a system which evolves spontaneously in a critical state characterized by spatial and temporal self-similarity. Dropping sand slowly, grain by grain, on a limited base, one reaches a situation in which the pile is critical, or has a critical slope. That means that a further addition of sand will produce slidings of sand (avalanches) that can be small or cover the entire size of the system. In this case the critical state, characterized by scale-invariant distribution for the size and lifetime of the avalanches, represents the attractor for the dynamics reached without the need of tuning of any critical parameter.

Many results has been obtained on the side of the characterization of the critical state in terms of critical exponents, and in clarifying the nature of self-organization [5–7]. In this paper we address the problem from a different point of view in order to characterize the dynamical properties of a class of sandpile models. The estimate of a suitable Lyapunov exponent, which turns out to be lower than zero, and the randomness of the model, allow us to describe the structure of the attractor in the configuration space. A numerical analysis of the divergence rate of two initially close configurations allows us to clarify the problem of the definition of a predictability for such a system.

We will refer in particular to the so-called Zhang model [8], a continuous, non-Abelian, version of the Bak, Tang, and Wiesenfeld (BTW) model [4], defined on a d-dimensional lattice. The variable on each site E_i (interpretable as energy, sand, heat, mechanical stress, etc.) can vary continuously in the range [0,1] with the threshold fixed to $E_c=1$. The dynamics is the following: (a) we choose a site in a random way, and we add to this site an energy δ (rational or irrational); (b) if at a certain time t a site, say i, exceeds the threshold E_c, a relaxation process is triggered, defined as

$$E_{i+\mathrm{NN}} \rightarrow E_{i+\mathrm{NN}} + \frac{E_i}{2d}\ , \qquad E_i \rightarrow 0 \tag{1}$$

where NN indicates the $2d$ nearest neighbors of the site i; (c) we repeat point (b) until all the sites are relaxed; and (d) we go back to point (a). We can also define a deterministic version of this model in which, at each addition time, one increases the variable of every site of the quantity δ and then one follows the same rules of the random case.

The dynamics of this model, either in the random as in the deterministic case, can be seen as described by a piecewise linear map. In fact, indicating with $x \equiv \{x_i\}_{i\in D}$ the configuration of the system at a certain time, where $D \subset \mathbb{Z}^2$ is the bounded domain whose cardinality is $|D|=N^d$ with N being the linear dimension of the lattice, the operator corresponding to a toppling at site i is given by

$$\Delta_i : (\Delta_i x)_j = x_j - \delta_{i,j} x_i + \tfrac{1}{4} \sum_{i':\langle i,i'\rangle} \delta_{i',j} x_i\ , \tag{2}$$

where $\langle i,i'\rangle$ means that i and i' are nearest neighbors.

Equation (2) shows that the single toppling is a linear operator, and acts as a local Laplacian. The evolution of a configuration up to the time t can be written as [9]

$$x(t) = T^t x_0 = L_t x_0 + \delta \sum_{s=1}^{t} L_{t-s+1} 1_{k(s)}\ , \tag{3}$$

where L_t is a linear operator defined as a product of linear operator Δ as

$$L_t \equiv \prod_{s=1}^{t} A_{t-s+1}\ , \quad A_t \equiv \prod_{i=1}^{q(t)} \Delta_{j(t,i)}\ . \tag{4}$$

$q(t)$, in the expression of the avalanche operator A_t at the time t, is the number of topplings of the avalanche started at the time t and $j(t,i)$, the site in which the ith toppling of this avalanche occurs. In fact, in a single avalanche the evolution of a generic vector x is ruled by the expression

$$x \rightarrow \Delta_{j(t,q(t))} \Delta_{j(t,q(t)-1)} \cdots \Delta_{j(t,2)} \Delta_{j(t,1)} x \equiv \prod_{i=1}^{q(t)} \Delta_{j(t,i)} x\ . \tag{5}$$

By iterating this process for the entire avalanche up to the time t, we obtain (3) via Eq. (4). x_0 is the initial configuration, and 1_i is a vector in $\mathbb{R}^D$ whose component i is 1 and all the others are 0. $k(s)$ defines the sequence of sites over which there will be the random addition of energy at the time s. It is worth stressing how t indicates just the time of addition of energy, and every single avalanche is supposed to happen between two addition times.

In order to characterize the linear map (3) we can define the *Lyapunov exponent* for this system. If the dynamics is C^1, i.e., described by a continuous map with continuous derivative, the Lyapunov exponent corresponding to a given trajectory $x(t)=T^t x_0$ can be defined, linearizing the dynamics in the neighborhood of $x(t)$, as [10]

$$\lambda \equiv \lim_{t\to\infty} \sup \sup_{dx\in\mathbb{R}^{|D|}} \frac{1}{t} \ln \frac{|T^t dx|}{|dx|} , \tag{6}$$

where $|\;|$ is a norm in $\mathbb{R}^{|D|}$. In our case the dynamics, being described as a piecewise linear map, is not C^1. Nevertheless if the two trajectories $x(t)$ and $y(t)$ make the same sequence of topplings, Eq. (6) holds with the substitution $y-x \to dx$. In fact, in this case, it holds $T^t y - T^t x = T^t(y-x) = T^t dx$. Therefore, definition (6) for the Lyapunov exponent begin to fail when, at a certain time, the two configurations make different topplings. It is easy to see that such a situation occurs when, for one configuration, $x_i(t)=1$ holds for some i and t. In this case a slight difference in the second configuration $y_i(t)=x_i(t)+\epsilon$ will produce a toppling just in the y configuration. From this point onward the two configurations will follow different sequences, and definition (6) fails definitely. In the phase space we can then recognize a zero-measure set of "bad" points on which two configurations close each other at will, and will diverge definitely. We can then identify this set as

$$I \equiv \{x \equiv \{x_i\}_{i\in d}:\ \exists\ i,t:\ x_i(t)=1\} \tag{7}$$

We are going to discuss this point again below. It is easy to see that the Lyapunov exponent is not positive. In fact, the dynamics in the tangent space, the dynamics of a small difference between two configurations, follows the same rules of the usual dynamics, and the "error" is redistributed to the nearest neighbor site.

It is then clear that the distance between two configurations, given by $|y-x|=\sum_i |dx_i|$, where $dx_i=(y_i-x_i)$, being conserved in the topplings far from the boundaries, can just decrease when a site of the boundary topples. We can conclude that $\lambda \le 0$.

In [9] we demonstrate a theorem that states that the maximum Lyapunov exponent is strictly lower than 0 and, in particular

$$\lambda \le -\frac{1}{|D|[d(D)+1]^2(\ln|D|+1)(1/\delta+1)} , \tag{8}$$

where $d(D)$ is the diameter of the set D.

An immediate consequence of this theorem is that the dynamics, up to the time t (for t sufficiently large), is given by a *piecewise linear contractive map.*

This allows us to address the following problems: (i) What is the structure of the snapshot attractor? (ii) How is the attractor of this system? (iii) What can be said about the predictability of this system?

(i) A snapshot attractor is obtained by considering a cloud of initial conditions and letting it evolve forward in time under a given realization of the noisy dynamics. The resulting measure at a single instant is the snapshot attractor. It is easy to show [9] that, given a set A in the configuration space Γ, the evolution tends to contract the measure of this set. This means in particular that the snapshot attractor has measured zero. Two different mechanisms concur in the formation of the snapshot attractor: (a) a volume contraction mechanism due to the effect of the negative Lyapunov exponent, and (b) a splitting mechanism which maps single sets of configurations in two or more distinct sets that are also far apart in the phase space. The dynamical balance of these two effects represents a basis for the definition of a predictability for such a system.

(ii) The attractor of the system is obtained by looking at a single initial condition under a realization of the random dynamics and plotting, after a long transient period, its position, time by time. In order to study the attractor of this system we performed some numerical simulations, by considering a one-dimensional chain of sites driven in a random way.

In the random case a box-counting analysis [12] on a two-dimensional projection of the configuration space seems to show the existence of a fractal attractor with a dimension D_f for the recurrent set of configurations. Figure 1(a) shows the two-dimensional projection of the configuration space of a linear chain of $L=80$ sites. Figure 1(b) reports the result of the box-counting analysis of the set of Fig. 1(a).

It is worth stressing that, for each given starting configuration, or better, for each set of the configuration space whose points follow the same sequence of topplings [9], the dynamics is described by an iterated function system (IFS) [13]. In [13] it has been proved that, when the iterative function is a contractive linear function, it exists a unique invariant measure. Examples of geometrical objects generated by an IFS are the cantor set or the Sierpinsky gasket [13]. In our case the situation is more involved, as the linear map is piecewise. Nevertheless it seems reasonable that the attractor is fractal.

(iii) At this point we would like to address the problem of the definition of a predictability for these systems. At first one could think that the existence of a negative Lyapunov exponent should assure a perfect predictability. That is not true. What makes the situation complex is the splitting mechanism mentioned above. Let us again consider the evolution of two close configurations differing by a quantity ϵ. When the energy of a site in a configuration reaches exactly the threshold, no matter how little ϵ is, it could produce a different sequence of topplings in the two configurations. We may say that the smaller ϵ is in respect to the minimal distance $d_{\min}$ of whatever site from the threshold, the higher will be the probability that the configurations will follow the same sequence of topplings. We then have to compare the two rates of approach of two configurations, and of the approach to the threshold $E_c=1$ or, what is the same, to the set I.

Figure 2 shows the rate of approach to the set I. This is obtained by simulating the evolution of a generic configuration driven by adding an irrational quantum of

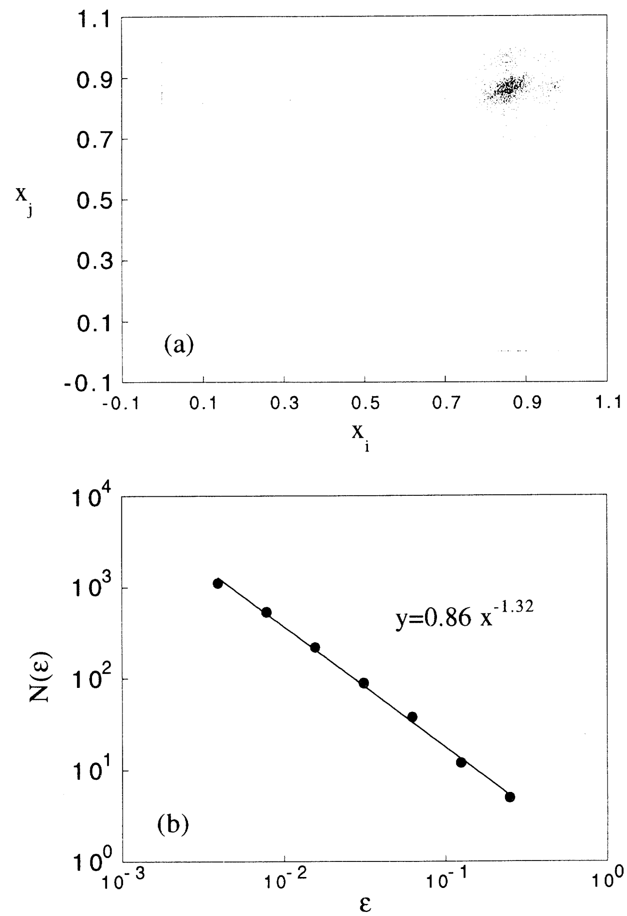

FIG. 1. (a) Two-dimensional projection of the configuration space for a one-dimensional chain of $L=80$ sites randomly driven; x_i and x_j indicate the coordinates on the generic plane on which the configuration space has been projected. (b) Box-counting analysis for the two-dimensional projection of the configuration space for a one-dimensional chain of $L=80$ sites.

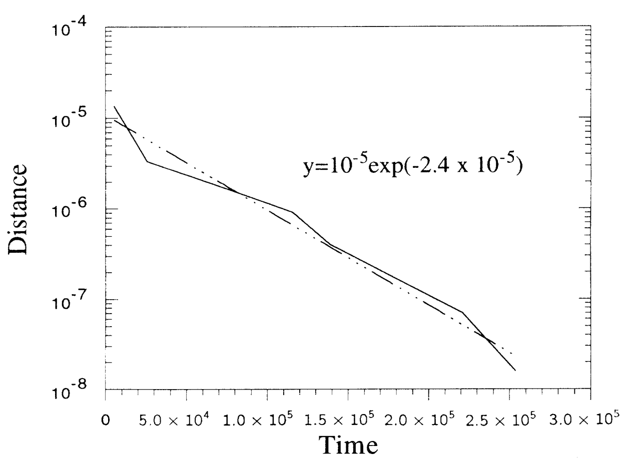

FIG. 2. Minimum distance of the system from the set I in the time interval $[0,t]$ (solid line). System of $L=80$ sites driven with the random addition to the irrational quantity $\delta=\sqrt{2}/8$. The dashed line indicates the numerical fit of the curve.

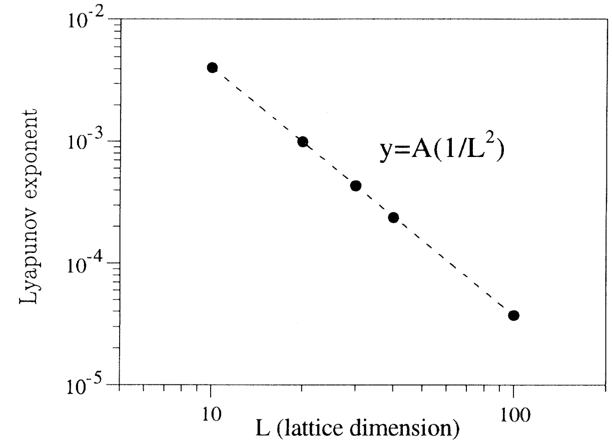

FIG. 3. Maximum Lyapunov exponent vs the lattice dimension.

energy ($\delta=\sqrt{2}/8$ in our case). The approach to the set of "bad" points is fitted by the curve $d\simeq d_0 e^{-\lambda' t}$, with $d_0\simeq 10^{-5}$ and $\lambda'=2.4\times 10^{-5}$, to be compared with the rate of approach of two different configurations which is given by $\epsilon(t)\simeq\epsilon(0)e^{\lambda t}$, where λ is the Lyapunov exponent defined in (6). Figure 3 shows the scaling of the Lyapunov exponent with the lattice dimension L. For the same L of Fig. 2, we have $|\lambda|=6\times 10^{-5}$. The fact

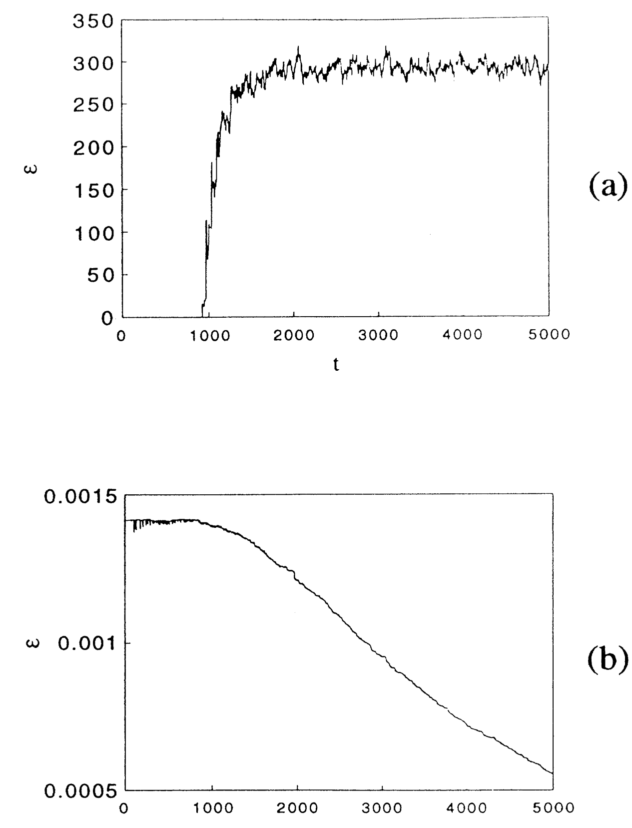

FIG. 4. Evolution of the distance ϵ between two configurations with a starting distance of (a) 10^{-2}, and (b) 10^{-3}.

that $|\lambda| > \lambda'$ suggests the existence of a sort of threshold in ϵ (ϵ_T).

In the random case the threshold has a probabilistic value. In fact if $\epsilon(0) < \epsilon_T$ we cannot exclude the possibility of different topplings, due to a too rapid approach to the set of "bad" points. The probability associated with this event become exponentially small as the time goes on because, while the system approaches the set I, at the same time, two different configurations tend to converge on each other.

In order to confirm these predictions, we simulated the parallel evolution of two different configurations in the random case (for a system with $L = 30$) with different starting error ϵ, and we plotted the distance (in the L^1 norm) between the two orbits. The results, shown in Figs. 4(a) and 4(b), respectively, for $\epsilon = 10^{-2}$ and 10^{-3} seem to confirm the existence of a probabilistic threshold in ϵ which determines the divergence or the asymptotic convergence of two orbits.

These results place the problem of the definition of a predictability into a wider perspective in which the Lyapunov exponent is not the only relevant quantity. Up to the time in which two different configurations make the same sequence of topplings, the error ϵ will decrease and the system remains predictable. From this point onward the evolution of the distance between the two configurations seems far from being linked to the Lyapunov exponent. The threshold mechanism, and then the splitting mechanism, therefore play crucial roles in determining the predictability of such systems.

We just gave an example in the case in which two configurations are driven by the same noise. It is very interesting, from the point of view of the predictability, to investigate what happens when two configurations are driven by different noises [14].

In conclusion we showed, for a particular class of sandpile models, the existence of a negative Lyapunov exponent which allows us to describe the system as a piecewise linear contractive map. We studied the structure of the attractor for the dynamics of this kind of map, showing the different mechanisms concurring with its formation and its low dimensionality. In the random case the system seems to live asymptotically on a fractal attractor, which can be put in relation with the attractor of an iterated function system. Finally, we showed how the predictability for such a class of models is related to a threshold mechanism, in which the Lyapunov exponent is not the only relevant quantity.

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[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).
[2] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981).
[3] L. Pietronero and H. J. Wiesmann, J. Stat. Phys. **36**, 811 (1984).
[4] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[5] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989); D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[6] P. Grassberger and S. S. Manna, J. Phys. I France **51**, 1077 (1990).
[7] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994); A. Vespignani, S. Zapperi, and L. Pietronero, Phys. Rev. E **51**, 1711 (1995).
[8] Y. C. Zhang, Phys. Rev. Lett. **63**, 470 (1987).
[9] E. Caglioti and V. Loreto (unpublished).
[10] G. Benettin, L. Galgani, and J. M. Strelcyn, Phys. Rev. A **14**, 2338 (1976).
[11] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[12] T. Vicsek, *Fractal Growth Phenomena* (World Scientific, Singapore, 1989).
[13] M. F. Barnsley, *Fractals Everywhere* (Academic, London, 1988).
[14] V. Loreto, G. Paladin, and A. Vulpiani, Phys. Rev. E (to be published).