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Characterization of chaos in random maps

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Abstract

We discuss the characterization of chaotic behaviors in random maps both in terms of the Lyapunov exponent and of the spectral properties of the Perron–Frobenius operator. In particular, we study a logistic map where the control parameter is extracted at random at each time step by considering finite-dimensional approximation of the Perron–Frobenius operator.

1. Introduction

The study of systems with many degrees of freedom is one of the central problems in the field of dynamical systems [1]. In many cases, there exists a separation of the time scales, i.e. a fast evolution superimposed to a slow one, that allows one to capture the dynamics in terms of simple models given by one-dimensional random maps [2]. For instance, in the fault dynamics, a fast evolution on small scales coexists with a slow one on geological times [3].

In this paper we shall consider random maps where the fast evolution is taken into account by choosing at random at each time step a particular deterministic one-dimensional map, while the slow evolution is given by the iteration of the map extracted time by time. Following the ideas of Spiegel et al. [2], we start with a system made of two variables x_n (representing the slow degrees of freedom) and y_n (the fast ones), evolving according to the deterministic rule

$$\begin{aligned} y_{n+1} &= g_1(y_n)\,, \\ x_{n+1} &= g_2(x_n, y_n)\,. \end{aligned} \tag{1}$$

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If the typical evolution time of y_n is much shorter than the corresponding time for x_n, such a model can be approximated in terms of the random map

$$x_{n+1} = f^{(i_n)}(x_n)\,, \tag{2}$$

where the integer $i_n = 1, \ldots, k$ is, e.g., an independent random variable, extracted at each time step with probability $p_1, \ldots, p_n$ and $f^{(1)}, \ldots, f^{(k)}$ are appropriate one-dimensional maps of the interval $[0, 1]$ into itself. This kind of random maps have been used to describe many models such as magnetic dynamos [2], transport in fluid [4], random systems [5] and on–off intermittency (abrupt switching among regular and chaotic behaviors) [2]. Moreover, random maps exhibit many interesting features such as scaling laws for the probability of the laminar phases [2].

The characterization of the behavior generated by random maps requires the extension of the tools used for the description of deterministic chaos from a physical point of view, and the extension of the rigorous results obtained for expanding maps from a mathematical one. In our opinion, both these problems are technically non-trivial, and worth to be analyzed because of the physical relevance of the phenomena involved.

The outline of the paper is the following. In Section 2 we introduce the basic concepts which enter in the characterization of random maps. In particular, we discuss the *average probability distribution* which is the analogue of the invariant measure for deterministic dynamical systems, and the so-called snapshot attractor. It is also given the definition of the Lyapunov exponent and of complexity for random dynamical systems. Section 3 is devoted to the Perron–Frobenius operator for random maps. In Section 4 we study the case of a random logistic map analyzing the three different regimes exhibited by the system for different values of the randomness parameter p. We also try to get a deeper understanding of the dynamics by using a finite-dimensional approximation of the Perron–Frobenius operator. Finally, in Section 5 we draw the conclusions and indicate the possible drawbacks of the Perron–Frobenius operator approach to the study of random dynamical systems.

2. Basic tools to study random maps

Let us briefly summarize the basic concepts needed to characterize random maps.

2.1. *The average probability distribution*

After a transient, a deterministic map evolves on the invariant set of the dynamics (usually an attractor), where it is possible to define an invariant probability measure. We should now consider the probability density obtained as a limit of the histogram of points given by the iterations of (2), on the 'average' invariant set I. It is natural to expect that such a distribution might be obtained by means of an average over the randomness realizations. Let us recall that for a deterministic map f, the invariant probability measure ρ is the eigenfunction of the Perron–Frobenius operator L related to the maximum (in modulus) eigenvalue $\gamma_1 = 1$, i.e.,

$$\rho(x) = L\rho(x) . \tag{3}$$

The operator is defined as follows:

$$L\phi(x) = \int_I dy\, \delta(y - f(x))\phi(y) = \sum_{z=f^{-1}(x)} \frac{\phi(z)}{|f'(z)|} . \tag{4}$$

A moment of reflection shows that for a random map it is still possible to find the average probability density by the straightforward generalization of Eq. (3):

$$\rho_{av}(x) = \overline{L}\rho_{av}(x) , \tag{5}$$

where we have introduced the (annealed) average operator

$$\overline{L} = \sum_{j=1}^{k} p_j L_j \tag{6}$$

and L_j is the Perron–Frobenius operator associated to the deterministic map $f^{(j)}$.

2.2. The snapshot attractor

In defining the average probability density, we have considered the long-time evolution of a single trajectory under a 'typical' realization of the randomness. One can also consider the probability density obtained by starting with a cloud of initial conditions $x_0^{(1)}, \dots, x_0^{(M)}$, with M very large, that evolve in time under *the same* randomness realization of the dynamics. There exists therefore a instantaneous probability density $\rho_t(x)$ obtained by the histogram of the $x_t^{(j)}$ in the limit $M \to \infty$. Such a probability measure is usually indicated as the snapshot attractor [4]. It is generally different of the average probability density, and asymptotically there are the following situations:

(I) $\rho_t(x) = \delta(x - x^*) = \rho_{av}(x)$

where x^* is a stable fixed point:

(II) $\rho_t(x) = \delta(x - x_t^*)$

where x_t^* changes in time and $\rho_{av}(x)$ is not a delta function:

(III) $\rho_t(x) = g(x,t)$

where $g(x,t)$ is a non-trivial function of the space–time.

Regimes (II) and (III) can be characterized by studying other quantities such as Lyapunov exponents, time correlations and complexity measures for random systems, see e.g. Ref. [6].

2.3. The Lyapunov exponent for random maps

In noisy systems, the Lyapunov exponent λ_I provides the simplest information about chaoticity and can be computed considering the separation of two nearby trajectories evolving in the same realization of the random process $I(t) = i_1, i_2, \ldots, i_t$. It is possible to introduce the Lyapunov exponent for random maps by considering the tangent vector evolution:

$$z_{n+1} = \left.\frac{df^{(i_n)}}{dx}\right|_{x_n} z_n \,, \tag{7}$$

so that for almost all initial conditions

$$\lambda_I = \lim_{N\to\infty} \frac{1}{N} \ln |z_N| \,. \tag{8}$$

It is worth to stress that this characterization of the chaoticity of a random dynamical system can be misleading [6, 7]. A negative value of the Lyapunov exponent computed in such a way implies predictability ONLY IF the realization of the randomness is known. In other more realistic cases of trajectories initially very close and evolving under different realization of the randomness, it can happen that after a certain time the two trajectories will be very distant even with a negative Lyapunov exponent λ_I.

Both in regimes (I) and (II), the Lyapunov exponent is negative, and in phase (II) it also corresponds to the typical contraction rate of a cloud of points toward the 'jumping' snapshot attracting point x_t^*, i.e., one has

$$\sigma_t^2 = \lim_{M\to\infty} \frac{1}{M} \sum_{j=1}^{M} \left(x_t^{(j)} - \frac{1}{M}\sum_{i=1}^{M} x_t^{(i)} \right)^2 \sim e^{-2|\lambda|t} \,. \tag{9}$$

On the other hand, we expect $\lambda > 0$ in phase (III).

2.4. Measures of complexity

We pointed out how the characterization of the chaoticity of a random dynamical system by the Lyapunov exponent can be misleading. On the other hand, it is possible, for such a system, to introduce a measure of complexity K which better accounts for their chaotic properties [6, 7], as

$$K \simeq h_s + \lambda_I \theta(\lambda_I) \,, \tag{10}$$

where h_s is the Shannon entropy [8] of the random sequence $I(t)$, λ_I is the Lyapunov exponent defined above and θ is the Heaveside step function. The meaning of the complexity K is rather clear: $K \log 2$ is the mean number of bits, for each iteration, necessary to specify the sequence $x_1, \ldots, x_t$ with a certain tolerance Δ.

We stress again that a negative value of λ_I does not implies predictability. To illustrate this point, let us calculate K for a system described by a random map which exhibits the so-called *on–off* intermittency [2] (see also Section 4). In this case, one has laminar phases, i.e. $x_t \simeq 0$, of average length l_L and intermittent phases of average length l_I. It is easy to realize that for $l_I \ll l_L$

$$K \simeq \frac{l_I}{l_L} h_s \tag{11}$$

since one has just to compute the contributions of the intermittent bursts whose relative weight is $l_I/(l_I + l_L) \simeq l_I/l_L$.

3. Perron–Frobenius operator for random maps

A good understanding of random maps can be achieved by studying the spectral properties of the Perron–Frobenius operator. In practice, it is convenient to treat the problem via finite-dimensional approximations. We have that each deterministic map $f^{(j)}(x)$ $(j = 1,\ldots,k)$ defines a Perron–Frobenius operator L_j via (4). The evolution of the average density is then obtained by the recursive relation

$$\rho_{av}^t(x) = \sum_{j=1}^{k} p_j L_j \rho_{av}^{t-1}(x)\,. \tag{12}$$

Requiring that ρ_{av} is stationary in time, we see that ρ_{av} is the eigenfunction of the average Perron–Frobenius operator corresponding to the maximum eigenvalue 1, that is,

$$\rho_{av}(x) = \sum_{j=1}^{k} p_j L_j \rho_{av}(x)\,. \tag{13}$$

The Lyapunov exponents can then be obtained by a space average over the attractor as

$$\lambda_I = \int_I \sum_{j=1}^{k} p_j \ln \left| \frac{df^{(j)}}{dx} \right| \rho_{av}(x)\, dx\,. \tag{14}$$

Moreover, the spectrum of the Perron–Frobenius operator is directly related to the decay rates of the time correlations also called 'resonances' of the dynamical system [9]. In particular, if we consider a generic smooth observable A, the time correlation at large times decays as

$$C_A(t) = \langle A(x_t)A(x_0)\rangle - \langle A\rangle^2 \sim e^{-\gamma t}\,, \tag{15}$$

where $\gamma > 0$, $\exp(-\gamma)$ is the modulus of the second eigenvalue of the Perron–Frobenius operator and the average is a time-average,

$$\langle A\rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{i=1}^{T} A(x_i)\,. \tag{16}$$

For random maps γ is given by the logarithm of the modulus of the second eigenvalue of the average Perron–Frobenius operator $\overline{L}$.

Beyond the average operator, it is interesting to consider the product of random transfer operators operator

$$\mathscr{P}_t(i_1,\ldots,i_t) = L_{i_t} L_{i_{t-1}} \ldots L_{i_1} \tag{17}$$

that is itself a random operator acting in an appropriate functional space. Roughly speaking, it can be regarded as the product of infinite-dimensional matrices whose elements are transition probabilities. It is therefore natural to expect that in the limit $t \to \infty$, $(\mathscr{P}_t \mathscr{P}_t^+)^{1/2t}$ for almost all the randomness realizations $\{i_1, \ldots, i_t\}$, has a non-random spectrum, as it happens for the product of random matrices, where the Oseledec theorem [10] holds. This result has been proved, under additional assumptions, in the mathematical literature in the case of infinite-dimensional operators acting on Banach spaces [11]. In addition, Ledrappier and Young [12] showed that there exists what could be called a random strange attractor, i.e. there exists $\rho_{av}(x)$, for the composition of independent random diffeomorphisms when the largest Lyapunov exponent is positive. Moreover, they proved the validity of a generalizied Pesin equality between the sum of the positive Lyapunov exponents and the Kolmogorov–Sinai entropy, when the stationary measure is smooth without requiring that the maps individually preserve a smooth measure.

The operator $\mathscr{P}_t$ is interesting since it controls the evolution of the instantaneous probability density $\rho_t(x)$:

$$\rho_t(x) = \mathscr{P}_t \rho_0(x) . \tag{18}$$

This problem is analogous to that one of the evolution of a tangent vector in dynamical systems, see Ref. [13]. In that case the jacobian matrix plays the role of the Perron–Frobenius operator and the tangent vector the role of the probability density. Note, however, that in the usual deterministic dynamical system problems the jacobian matrix is not extracted at random, although given by a chaotic dynamics.

The first eigenvalue of $\mathscr{P}$ is equal to unity since $\rho_t(x)$ is a normalized density. The second eigenvalue $\tilde{\gamma}_2 < 1$ measures the exponential rate of collapse of a generic density ρ_t toward the time-dependent eigenfunction $\tilde{\rho}_t$. In other terms in an appropriate norm, one has

$$\|\rho_t - \tilde{\rho}_t\| \sim e^{t \ln |\tilde{\gamma}_2|} . \tag{19}$$

It is worth stressing that in the phase (B) where the attractor is the jumping point x_t^*, $\ln |\tilde{\gamma}_2| = \lambda$ measures the contraction rate of a cloud centered around x_t^*. On the contrary, in the fully chaotic phase (C), $\ln |\tilde{\lambda}_2|$ has no particular intuitive meaning.

4. Transition to chaos in the random logistic map

In this section we analyze a specific example of random map, namely,

$$x_{t+1} = r_{i_t} x(1 - x) , \tag{20}$$

with

$$r_{i_t} = \begin{cases} 4 & \text{with probability } p , \\ \frac{1}{2} & \text{with probability } 1 - p . \end{cases} \tag{21}$$

The evolution is thus given by the random composition of a contracting and of an expanding logistic map. For sake of simplicity the expanding map has the control parameter set at the Ulam point ($r = 4$). Indeed, for $p = 1$, it is possible to find by a well-known duality argument the invariant probability density,

$$\rho(x) = \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}} . \tag{22}$$

At varying p, one observes three different regimes:

(1) $p < \frac{1}{3}$. The random map has a stable fixed point in the origin. The average probability density $\rho_{av}(x) = \delta(x)$ and the Lyapunov exponent is negative and it is trivially given by the expression

$$\lambda^{(1)} \cdot p + \lambda^{(2)} \cdot (1-p) , \tag{23}$$

where $\lambda^{(1)} = 4$ and $\lambda^{(2)} = \frac{1}{2}$ are the Lyapunov exponents of the two logistic maps (21).

(2) $\frac{1}{3} < p < p_c = 0.47\ldots$. There is a snapshot attractive fixed point x_t^* and $\rho_{av}(x) \neq 0$ on all the interval $[0,1]$ while $\rho_t(x) = \delta(x - x_t^*)$. The Lyapunov exponent is still negative and at least near the transition the complexity is given by the expression already cited in Section 2.

$$K \simeq \frac{l_I}{l_L} h_s , \tag{24}$$

where l_L and l_I are, respectively, the average lengths of the laminar and intermittent phases and the (24) is written in the case $l_I \ll l_L$ (see Fig. 1).

(3) $p > p_c$.

The random map is chaotic: the Lyapunov exponent is positive, both $\rho_{av}(x) \neq 0$ and $\rho_t(x) \neq 0$ on all the interval $[0,1]$.

We have tried to get a deeper understanding on the dynamics by considering finite-dimensional approximations of the Perron–Frobenius operator. The idea is to treat the two Perron–Frobenius operators L_1 and L_2 related to the contracting and to the expanding map as $N \times N$ matrices. In fact, there are two possible strategies.

The first one is to consider a uniform partition of the interval $[0,1]$ made of N subintervals. Then we approximate L by the matrix of the transition probabilities between the partition elements. In the limit $N \to \infty$ such a matrix converges to L according the Ulam conjecture [14], that can be proved in some cases [15, 16].

The second one is to look for the matrix representation of the operator in an appropriate base of the functional space of polynomials up to order $(N-1)$, i.e. the so-called Galerkin approximation [17].

We have used both the strategies, even if it turns out that the Ulam approximation is in practice more convenient. The Galerkin approximation using Legendre polynomials gives unavoidable numerical troubles since the coefficients of the powers grow very fast with the order n of the polynomial.

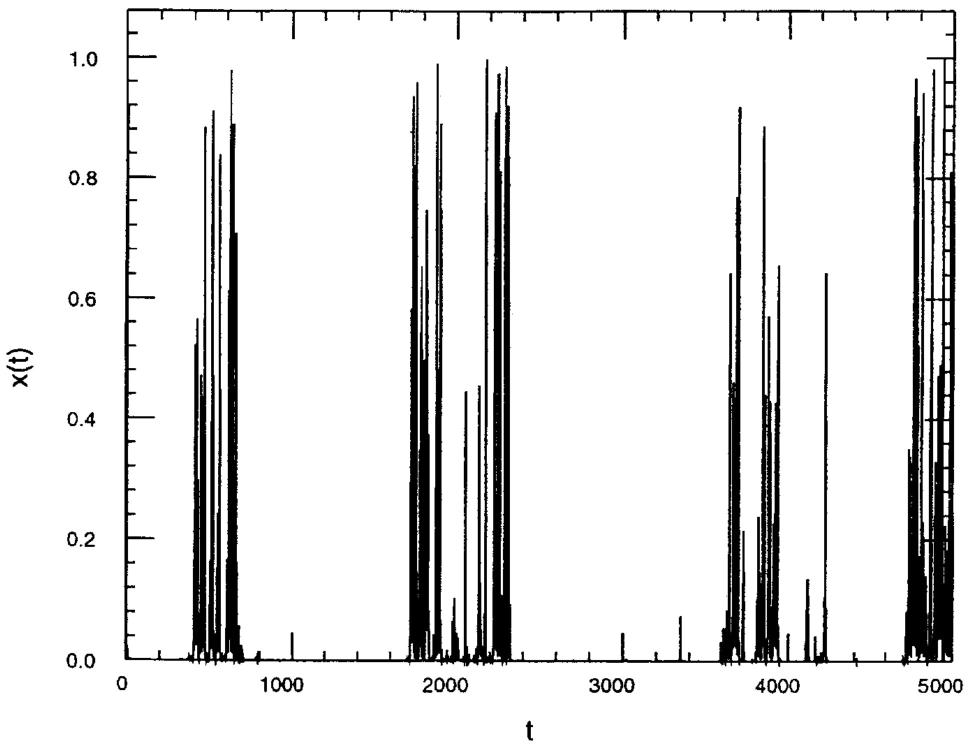

Fig. 1. x_t vs. t for the random logistic map described by Eqs. (21) with $p = 0.35$.

We performed the numerical treatment of the Perron–Frobenius operator as follows. For a given partition of the interval $(0,1)$ in N segments the Perron–Frobenius operator L_α ($\alpha = 1,2$) of the map α is approximated by the $N \times N$ matrix $\hat{L}_\alpha(N)$ whose elements $[L_\alpha(N)]_{i,j}$ are given by the probability to perform a transition from the interval j to the interval i and $\overline{L}$ is the matrix

$$\overline{L}(N) = pL_1(N) + (1-p)L_2(N). \tag{25}$$

From (25) one can easily compute $\rho_{av}(x)$, λ_I and γ.

Fig. 2 shows λ_I vs. p. One can observe how the Lyapunov exponent computed using the approximated Perron–Frobenius operator (25) is in good agreement with the exact (numerical) results only in region (III) and for very small values of p. On the contrary the finite dimension approximation of the Perron–Frobenius operator seems to have (if any) a very slow convergence in region (II).

Fig. 3 shows $C(\tau)$ vs. τ at different values of p. One roughly has for $p = p_c + \varepsilon$

$$C(\tau) \sim e^{-\gamma(\varepsilon)\tau}\tau^{-\alpha(\varepsilon)}, \tag{26}$$

where $\gamma(\varepsilon) \to 0$ for $\varepsilon \to 0$. This behavior recalls a critical phenomenon and it is essentially understood [2]. The approximation (25) of the Perron–Frobenius operator gives very bad results for γ.

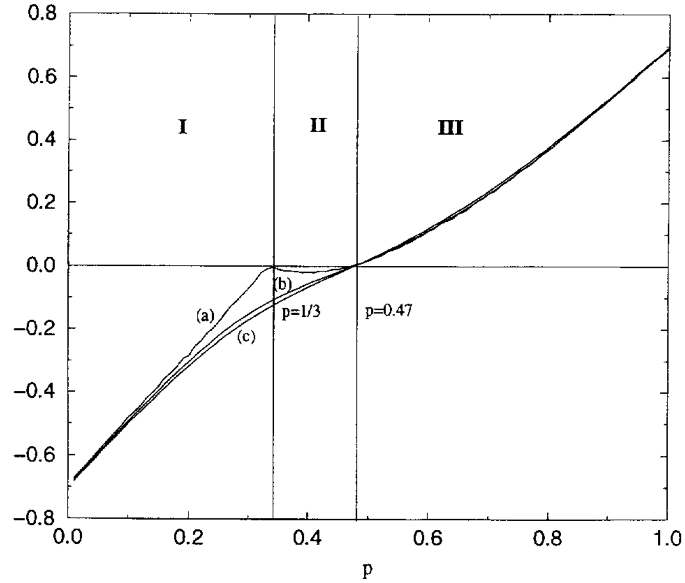

Fig. 2. Lyapunov exponent λ_I vs. p for the logistic random map described by Eqs. (21); (a) numerical result. Curves (b) and (c) indicate the theoretical estimation of the Lyapunov exponent with the Ulam method with matrices $N \times N$ with (b): $N = 200$ and (c): $N = 100$.

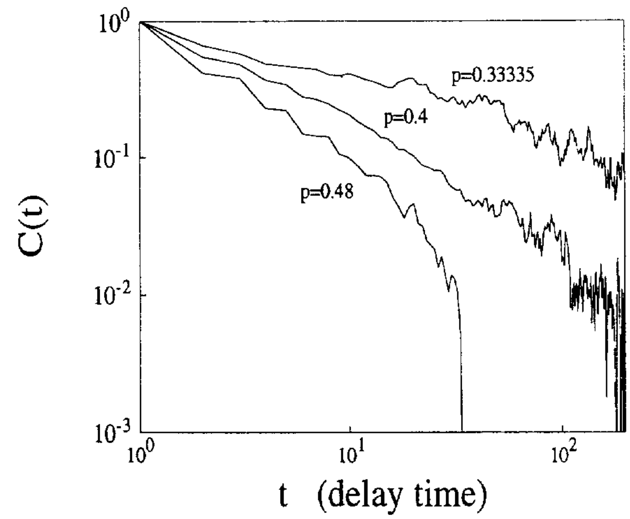

Fig. 3. Temporal autocorrelation functions for the random logistic map (21) for three different values of p.

Fig. 4 shows $\rho_{av}(x)$ vs. x for two different values of p. In region (II) for $p = p_c + \varepsilon$ one has a power-law behavior

$$\rho_{av}(x) \sim x^{-a(\varepsilon)} \,. \tag{27}$$

The approximation of the Perron–Frobenius operator is able to give a good approximation in regions (II) and (III).

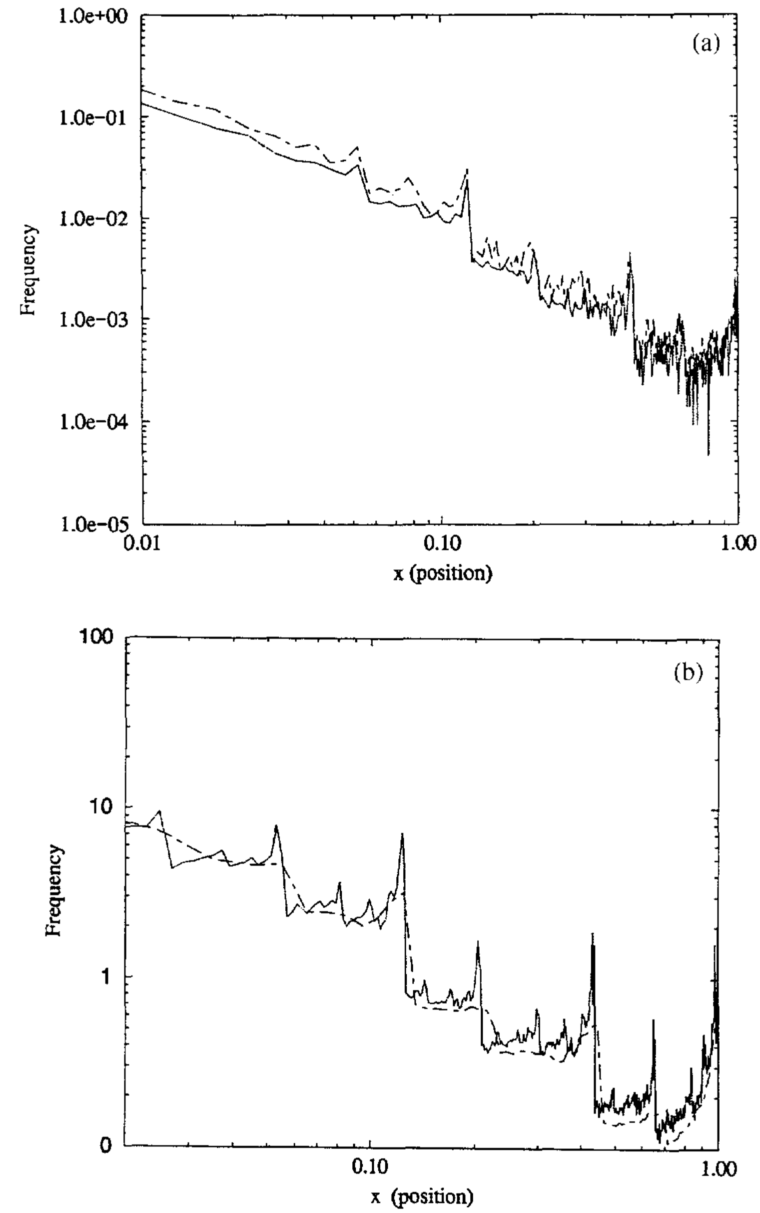

Fig. 4. Histograms of the distribution of positions for the random logistic map. The dot-dashed line corresponds to the theoretical estimation and the solid line to the numerical result: (a) $p = 0.35$; and (b) $p = 0.48$.

5. Conclusions

In this paper we have discussed the dynamical characterization of systems whose time evolution is described by random maps, with particular emphasis on the notion of Lyapunov exponent and of complexity. We have studied the application of the Perron–Frobenius operator to the analysis of the dynamical behavior of random dynamical systems. The Perron–Frobenius operator can be analyzed through finite-dimensional approximations in order to derive the invariant measure of the system, the Lyapunov exponent and the temporal correlation functions. In particular, we have followed two ways.

The first way consists in approximating of the Perron–Frobenius operator as an $N \times N$ matrix whose elements are the transition probabilities between the elements of the partition of the interval $[0,1]$ in N subintervals. According to the Ulam conjecture such a matrix should converge to the Perron–Frobenius operator in the limit $N \to \infty$. The second approach is the so-called Galerkin approximation and consists in the matrix representation of the operator in an appropriate basis of the functional space [17]. We found out that the first strategy gives the better results. In particular, it provides a good approximation of the average invariant measure ρ_{av} (see Figs. 4(a) and (b). As for the value of the Lyapunov exponent λ_I, the method is suitable to describe its behavior near the transition to chaos (from region (II) to region (III)) but the results are less good near the first transition between regions (I) and (II). In this case, in fact, the contributions to

$$\lambda_I = \sum p_j \int_I \ln \left| \frac{df^{(j)}}{dx} \right| \rho_{av}(x)\, dx$$

could be large exactly where the Perron–Frobenius method is not able to give an accurate approximation for ρ_{av}. Fig. 2 shows that at increasing the order of approximations of the Perron–Frobenius operator from a 100×100 to a 200×200 matrix, there is a tendance to obtain better and better approximation of the exact value of λ_I. We thus expect that in the limit $N \to \infty$ this approach tends to recover the real value of λ_I. As for the temporal correlation functions the method based on the approximation of the Perron–Frobenius operator does not work due, probably, to a too slow convergency with N.

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