# Dynamically Driven Renormalization Group

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**Abstract**

We present a detailed discussion of a novel dynamical renormalization group scheme: the Dynamically Driven Renormalization Group (DDRG). This is a general renormalization method developed for dynamical systems with non-equilibrium critical steady-state. The method is based on a real space renormalization scheme driven by a dynamical steady-state condition which acts as a feedback on the transformation equations. This approach has been applied to open non-linear systems such as self-organized critical phenomena, and it allows the analytical evaluation of scaling dimensions and critical exponents. Equilibrium models at the critical point can also be considered. The explicit application to some models and the corresponding results are discussed.

**Key words:** Renormalization group; non-equilibrium steady states; driven dynamical systems; self-organized criticality.

# 1 Introduction

The study of second order phase transitions introduced in the field of statistical physics the concept of criticality [1, 2, 3, 4]. A critical system is characterized by the absence of a characteristic lenghtscale: the system fluctuates strongly and correlation functions show non-analytic behavior. In the critical region therefore the usual tools of a physicist, principally perturbation theory, fail completely [4]. The renormalization group (RG) theory provides a comprehensive understanding of second order phase transition and critical phenomena [5, 6, 7, 8]. This theory has been a major breakthrough in statistical physics and lead to the study of many others scale invariant and critical phenomena. Among these phase transitions associated with non-equilibrium states [9, 10], fractal growth phenomena [11], self-organized critical (SOC) systems [12, 13] and a vast class of complex systems in which information spread over a wide range of length and time scales [14].

Since both areas of critical and complex systems deal with self-similar structures, it was natural for physicists who were familiar with RG techniques to consider these new problems as a possible playground for these methods. The situation turned out to be more complex, since the properties of non-equilibrium critical phenomena are quite different from those of ordinary critical systems. For instance, there is no ergodic principle and in general it is not possible to assign a Boltzmann weight to a configuration. For the above reasons many authors concluded that these problems pose questions of new type for which it would be desirable to have a common theoretical framework [15]. In the past decade an intense activity has been devoted to a better understanding of these systems and theoretical methods are being developed [10, 16].

Recently we introduced a renormalization scheme [17] for sandpile models [13], that has later been applied [18] to forest-fire models [19, 20]. This approach deals with the critical properties of the system by introducing in the renormalization equations a dynamical steady state condition which provides the non-equilibrium stationary statistical weights to be used in the calculation. In this way it is possible to characterize the fixed point dynamics and to compute analytically the critical exponents.

Here we present the general formulation of this novel type of dynamical renormalization group: the Dynamically Driven Renormalization Group (DDRG), a general theoretical method for dynamical non-equilibrium systems with critical stationary state. The essential idea is to combine a real space renormalization group (RSRG) scheme with the dynamical steady-state condition which characterizes the stationary regime. The RG equations are driven by the steady state condition feedback from which we obtain the configurations approximate statistical weight to be used in the dynamical renormalization of the master equation. While the approximate stationary distribution neglects correlations, these are considered in the dynamical renormalization. Finer scale correlations are then included in the approximate stationary distribution at the new scale which is calculated through the steady state condition with renormalized dynamical parameters. These strategy gives an RG scheme which can be improved considering increasingly better approximations in both the dynamical renormalization and the stationary distribution.

For the sake of clarity we discuss extensively the explicit application of the method to some driven dissipative systems referring to the present general framework. In particular we report in full details the results obtained for the critical height sandpile model and the forest fire model. For this class of models the DDRG can be considered a general renormalization scheme which provides a new class of analytical tools for the study of the stationary critical state. The DDRG can also be applied to ordinary dynamical critical phenomena for which the stationary state is characterized by Gibbs distributions. In this case the DDRG can be directly compared with other RG approaches.

The paper is organized as follows: sec.2 introduces the concept of non-equilibrium steady-state and the approximate description in term of mean-field stationary conditions. Sec.3 presents in full detail the Dynamically Driven Renormalization Group. The basic recursion relations are obtained and the conceptual scheme is discussed as well as the approximations involved. Sec.4 shows the explicit application of the DDRG to sandpile and forest fire models. Sec.5 shows the application scheme to ordinary critical phenomena. Sec.6 is devoted to conclusions and perspectives. In the appendix A we discuss in detail the coarse graining of time.

# 2 Equilibrium and non-equilibrium steady states

The distribution over configuration space in equilibrium ensembles as well as many extended dissipative and nonlinear dynamical systems evolve in stationary states described by time independent probability distributions. The stationary state in some cases shows long range correlations and self-similar properties[1]. To describe these phenomena several models have been introduced: here we will consider models defined on a discrete $d$-dimensional lattice. To each site of the lattice is associated a variable $\sigma_i$ that can assume $q$ different values ($\sigma_i = 1, 2, 3, \cdots, q$). The subscript $i$ labels the lattice site. A dynamics characterized by a set of parameters[2] $\mu$ acts on these variables and defines the temporal evolution of the model. The system can be described by $P(\sigma, t)$, the probability that at the instant $t$ the system is in the state $\sigma \equiv \{\sigma_i\}$. This is the usual way to characterize physical ensembles for which we want to know the statistical distribution in phase space.

For stationary processes, the system is invariant under uniform time translations. Thus, the variables $\sigma_i(t + \tau)$ are statistically indistinguishable from the untranslated variables $\sigma_i(t)$. This implies that all single time averages are constants and there is a single time probability density $W(\sigma)$ independent of time. An ensemble of systems at thermodynamic equilibrium is, for example, stationary as well as an ensemble in which the component systems are maintained in a non-equilibrium steady-state, at least after the ensemble has been sufficiently aged. For equilibrium ensembles, the steady state statistical distribution is given by the Gibbs distribution. In this case the functional dependence of the equilibrium distribution function $W(\sigma)$ on the parameters[3] $\mu$ should be consistent with statistical mechanics and properly describe the static equilibrium properties of the system. In particular, the partition function and the equilibrium expectation values should show critical point singularities appropriate to the spatial dimensionality and the symmetry of the order parameter. For non-equilibrium ensembles in principle one does not know how to assign a statistical weight to a given non-equilibrium configuration. An additional problem is that, since the system

[1] In equilibrium ensembles this happens for special values of the control parameter, i.e., at critical points

[2] We should use a vector notation $\boldsymbol{\mu} \equiv \{\mu\}$ to denote the set of all variables. For the sake of simplicity we use the simpler notation $\mu$.

[3] For instance the temperature or the set of applied fields.

is not in equilibrium, the distribution changes in time.

One possibility is to study systems which have "settled down" into non-equilibrium steady states, so that the distributions, while non Gibbsian, have become stationary. In this case we can describe, at least approximately, the single-time statistical distribution [21], since the densities $\rho_\kappa$ of sites in a particular state $\sigma_i = \kappa$ ($\kappa = 1, \cdots, q$) do not change, on the average, as a function of time. Associated to the set of time independent average densities $\{\langle\rho_\kappa\rangle\}$ there is a unique stationary probability distribution, that characterizes single-time averages in the steady-state ensemble. We can therefore characterize the average statistical state by means of stationarity conditions for the system. These can be obtained from mean field equations of the form

$$\frac{\partial}{\partial t}\{\langle\rho_\kappa\rangle\} = \mathcal{S}_\mu(\{\langle\rho_\kappa\rangle\}) \tag{1}$$

where the operator $\mathcal{S}_\mu$ describes the evolution of the system as a function of the dynamical parameters defined above. For example, in the case of dilute gases the non-linear function that gives the rate of change of the density becomes a non-linear functional, i.e. the collision operator. In general, the operator $\mathcal{S}_\mu$ represents the sum of dissipative contributions, mechanical (non-dissipative) contribution, and the effect of other external fluxes. Note that the operator on the left-hand side of Eq. (1) is time independent, so that the differential equation is first order in time.

Time independent solutions of Eq. (1) will be referred to as "steady-states", although we should keep in mind that those are only the average states of the ensemble. Mathematically the steady-states $\langle\rho_\kappa\rangle$ are determined by the equation

$$\frac{\partial}{\partial t}\{\langle\rho_\kappa\rangle\} = \mathcal{S}_\mu(\{\langle\rho_\kappa\rangle\}) = 0 \tag{2}$$

The above equation can have more than one solution, even when $\mathcal{S}_\mu$ is a linear function. In the following, however, we will consider only the presence of a single meaningful stationary state. In ordinary statistical systems, the Eq. 2 represents the thermodynamic equilibrium condition. For driven dynamical systems, it describes the *driving* of the system to the non-equilibrium steady state, by means of a balance condition. We can express the stability or self-organization properties in mathematical form as

$$\lim_{t\to\infty}\langle\rho_\kappa(t)\rangle = \langle\rho_\kappa\rangle \tag{3}$$

for any initial condition of the Eq.(1): the steady-state is an attractor for the dynamics.

It is possible to go further on these lines to find a more accurate description of the steady-state. For example, one could write down mean-field equations as Eq.(2) for the average density correlations. We are, however, interested in systems which show *critical properties* in the stationary state. This implies non-analyticity and long-range correlations, so that it is impossible to go much further along with a mean-field description. To describe critical systems and scaling behavior, we have to turn our attention to renormalization group methods.

# 3 The Dynamically Driven Renormalization Group

In this section we present the general formalism of a new type of dynamical renormalization group especially suited for systems with non-equilibrium critical steady-state. In fact, by using the dynamical steady-state condition we are able to develop a renormalization strategy which allows us to compute critical exponents in a wide range of non-equilibrium systems. The method can also be applied to equilibrium dynamical critical phenomena for which the steady-state distribution is given by the equilibrium Gibbs distribution.

In the following we will refer to RSRG schemes. The real-space formulation of the renormalization group, by virtue of its simplicity and versatility is a vital tool in the theoretical understanding of critical phenomena. In contrast to momentum-space renormalization group, the RSRG, in general does not have any systematic way to treat the approximations: there is no small parameter that controls an expansion. On the other hand, to improve the accuracy of the method one can use "higher order" techniques like the introduction of proliferation (additional couplings), larger cells, toroidal or rectangular transformation or other extrapolation scheme [7, 8, 22]. In this way, the critical exponents for several models have been obtained with good precision.

### 3.1 Coarse graining and renormalization

The essential ideas of the dynamic real space renormalization group approach derive from Kadanoff's block analysis [23] and from coarse graining of time proposed by Suzuki [24]. We begin summarizing the main derivation of the method. We start with the following general Master Equation (ME):

$$\frac{\partial}{\partial t} P(\sigma, t) = \mathcal{L}(\mu) P(\sigma, t) \tag{4}$$

where $P(\sigma, t)$ denotes the probability distribution function for the configurations of the system at time $t$, and $\mathcal{L}(\mu)$ is the temporal evolution operator, characterized by a set of dynamical parameters $\mu$. With the operator $\mathcal{R}$ we indicate the coarse graining operator, that eliminates degrees of freedom inside a cell and rescales time and space. The application of $\mathcal{R}$ yields

$$\mathcal{R}P(\sigma, t) = P'(S, t') \tag{5}$$

where $P'(S, t')$ denotes the probability distribution for the coarse grained system. More explicitly, we can write:

$$\mathcal{R}P(\sigma, t) = \mathcal{R}(e^{t\mathcal{L}} P(\sigma, 0)) = e^{t'\mathcal{L}'} P'(S, 0) \tag{6}$$

The scale transformation $t\mathcal{L} \to t'\mathcal{L}'$ yields the dynamical RG approach [25], while the scale transformation of $P(\sigma, 0)$ corresponds to the usual static RG approach [6, 7]. Denoting with the vector $\mu$ the parameters of the system, the RG yields recursion relations:

$$\mu' = f(\mu) \quad \text{and} \quad t' = g(t, \mu) \simeq t\, g(\mu) \tag{7}$$

from which it is possible to calculate the fixed points and the critical exponents of the model. These equations are obtained from the renormalization procedure which impose that $P'(S, t')$ has the same functional form as $P(\sigma, t)$.

The most delicate problem in RSRG approaches is to take a partial elimination of degrees of freedom in Eq.s (4) and (5) [8]. To deal with this problem many approximate methods have been proposed [25, 26] and have been mostly applied to the kinetic Ising model [27].

Achaim and Kosterlitz [28] perform a Migdal-Kadanoff [29] transformation on the probability distribution function, assuming a functional form of the type:

$$P(\sigma, t) = e^{-H(\sigma) + h(t)} \tag{8}$$

and treating the time dependent part as a perturbation. In this way they study the relaxation of the probability density towards equilibrium. The dynamical critical exponent is extracted from the scaling of the slower relaxation mode. Suzuki et al.[30] obtain the coarse graining of time studying the dynamical equation for the first moment of the probability distribution (i.e. the magnetization). The equation is decimated in the Migdal-Kadanoff approximation and the equation of motion for the decimated modes are solved perturbatively. The time scaling is chosen so that it keeps the equations in the same form. A different formalism, suitable to study the properties of the model even far from the critical region, was developed by Mazenko et al.[31] The coarse graining operator and the time rescaling are chosen self-consistently in order to insure the Markoffian behavior of the renormalized spin flip operator. One then writes recursion relations for the two-point correlation functions from which the critical properties of the model are extracted.

To develop a RSRG method suitable for irreversible non-equilibrium systems we will consider a more explicit treatment of the master equation. In particular we focus our attention on the dynamics of discrete models on a lattice characterized by a set of lattice variables $\sigma \equiv \{\sigma_i\}$, each of them being a $q$-state variable (see previous section), and by the dynamical parameters $\mu$. In this case we can rewrite eq.(4) as

$$P(\sigma, t) = \sum_{\{\sigma^0\}} \langle \sigma \mid T(\mu) \mid \sigma^0 \rangle P(\sigma^0, 0) \tag{9}$$

where $\langle \sigma \mid T(\mu) \mid \sigma^0 \rangle$ is the transition probability from the configuration $\sigma^0 \equiv \{\sigma_i^0\}$ to the configuration $\sigma \equiv \{\sigma_i\}$ in a unit time step $t$. The symbol $\sum_{\{\sigma\}}$ will always mean a summation over all the configurations. The operator $T$ is the discrete counterpart of the operator $\mathcal{L}$.

We then coarse grain the system by rescaling lengths and time according to the transformation $x \to bx$ and $t \to b^z t$. The renormalization transformation can be constructed through a renormalization operator $\mathcal{R}(S, \sigma)$ that introduces the coarse grained variables set $S \equiv \{S_i\}$ and rescales the lengths of the system [26]. This operator contains all the information connecting the coarse-grained state with the original one. Not every choice of the operator $\mathcal{R}$ will lead to a meaningful transformation, and in constructing it one should be guided by physical insight. Moreover, the transformation should satisfy some general properties in order to respect the dimension and the symmetry of the internal space of the dynamical variables, i.e. the renormalized variables should be of the same kind as the original. In general, $\mathcal{R}$ is a projection operator with the properties:

$$\mathcal{R}(S,\sigma) \geq 0 \quad \text{for any} \quad S \equiv \{S_i\}, \sigma \equiv \{\sigma_i\} \tag{10}$$

and

$$\sum_{\{S\}} \mathcal{R}(S,\sigma) = 1. \tag{11}$$

These properties preserve the normalization condition of the renormalized distribution. The explicit form of the operator $\mathcal{R}$ will be defined case by case in the various application of the method. Usually, it corresponds to a block transformation in which lattice sites are grouped together in a super-site that defines the renormalized variables $S_i$ by means of a majority or spanning rule.

We subdivide the time step in intervals of the unitary time scale and we apply repeatedly the dynamical operator $T$, obtaining

$$P(\sigma, N) = \sum_{\{\sigma^0\}} \langle \sigma \mid T^N(\mu) \mid \sigma^0 \rangle P(\sigma^0, 0) \tag{12}$$

where $T^N$ denotes the application of the $T$ operator $N$ times. We can therefore write the eq(5) for the coarse graining of the system as follows:

$$P'(S,t') = \sum_{\{\sigma\}} \mathcal{R}(S,\sigma) \sum_{\{\sigma^0\}} \langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle P(\sigma^0, 0) \tag{13}$$

where we have included the application of the operator $\mathcal{R}$ and $t' = b^z t$. The meaning of $\langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle$ has to be defined explicitly. In the simplest case $b^z = N$ where $N$ is an integer number. Whenever is possible to define a continuous time evolution operator, $T^{b^z}$ can be found as an integration over time. In general, since we are often dealing with discrete time evolution, we have to consider the $T^{b^z}$ as an effective dynamical operator. We will specify a projection operator $\mathcal{D}$ for the dynamics, that samples only the paths which lead to an appropriate definition of the dynamical process at the coarse grained scale. Also in this case, as for the operator $\mathcal{R}$, spanning conditions or majority rules are obtained from physical considerations. The projection operator is chosen in such a way to preserve the form of the operator $T$ at every scale. In this way it is possible to define recursion relations for the dynamical parameters. In appendix A, we present in detail the definition of the dynamical projector operator and the explicit form of the effective operator $T^{b^z}$.

In order to define the RG transformation more clearly, eq. (13) can be written as:

$$P'(S,t') = \sum_{\{\sigma^0\}} \sum_{\{S^0\}} \mathcal{R}(S^0,\sigma^0) \sum_{\{\sigma\}} \mathcal{R}(S,\sigma) \langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle P(\sigma^0, 0) \tag{14}$$

where we used the properties $\sum_{\{S^0\}} \mathcal{R}(S^0,\sigma^0) = 1$. By multiplying and dividing each term by $P'(S^0,0) = \sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0)P(\sigma^0,0)$, and changing the order of summations we have

$$P'(S,t') = \\ \sum_{\{S^0\}} \left( \frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma)\langle \sigma | T^{b^z}(\mu) | \sigma^0 \rangle P(\sigma^0,0)}{\sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0)P(\sigma^0,0)} \right) P'(S^0,0) \tag{15}$$

which finally identifies the renormalized dynamical operator and yields the coarse grained master equation in the form

$$P'(S,t') = \sum_{\{S^0\}} \langle S \mid T'(\mu) \mid S^0 \rangle P'(S^0,0) \tag{16}$$

In the following, we apply this scheme to systems with a stationary distribution $P(\sigma, t \to \infty) = W(\sigma)$ given either by an equilibrium or a non-equilibrium steady-state. We can therefore study the renormalization of the dynamics for small deviations from the stationary state

$$P(\sigma,t) = W(\sigma) + \varepsilon h(\sigma,t) \tag{17}$$

where $W(\sigma)$ is the steady-state probability distribution and $\varepsilon$ is an expansion parameter for the non-stationary part of the distribution. In the lowest order approximation we can consider only the stationary part of the distribution in the renormalization of the evolution operator $T$, which, by comparing Eq.(15) and (16), is given by:

$$\langle S \mid T'(\mu) \mid S^0 \rangle = \\ \frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma)\langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle W(\sigma^0)}{\sum_{\{\sigma^0\}} \mathcal{R}(S^0,\sigma^0)W(\sigma^0)}. \tag{18}$$

This is the basic renormalization equation that defines the dynamical evolution operator for the coarse grained system. In principle one could also consider higher orders in $\varepsilon$ with an *Ansatz*[4] for the form of $h$. For the systems we will study the zero order approximation will give already non trivial results, since we are interested in the dynamics of the system once the steady state has been reached.

To understand intuitively the above transformation (Eq.18) we can consider that the operator $\mathcal{R}(S,\sigma)$ is a projection operator that selects only the configurations $\{\sigma_i\}$ which, at the coarse grained level, are mapped into the configuration $\{S_i\}$. The right-hand side term of the above equation can be read as follows. The operator $\mathcal{R}(S^0,\sigma^0)$ selects only the configurations which are renormalized in the starting configuration $\{S_i^0\}$, each of them with a relative weight given by $W(\sigma^0)$ normalized with the factor $\sum \mathcal{R}(S^0,\sigma^0)W(\sigma^0)$. For each of these configurations we compute the statistical weight of the evolution to the configuration $\{\sigma_i\}$, through the paths selected by the dynamical operator $T^{b^z}$ (see the appendix). Finally, we sum up only the contribution of the configurations $\{\sigma_i\}$ that renormalize in the configuration $\{S_i\}$, selected from the operator $\mathcal{R}(S,\sigma)$. In other words, the new dynamical operator $T'$ is defined as the sum of the statistical weight of all the paths that lead from any starting configuration that renormalizes in $\{S_i^0\}$, to any final configuration that renormalizes in $\{S_i\}$. Each path has then an additional weight given by the relative probability of the starting configuration $\{\sigma_i^0\}$. This last factor is the stationary statistical distribution, and preserve the normalization of the transition probability matrix $T'$. It is interesting to note that the stationary configurations distribution compares explicitly as a statistical weight in the renormalization equations.

The Eq.(18) is the basic renormalization equation from which the desired recursion relations are obtained. Imposing that the renormalized operator $T'$ has the same form of the operator $T$, i.e. $T'(\mu) = T(\mu')$, we obtain

[4] To first order in $\varepsilon$ one obtains:

$$h'(S,t') = \sum_{\{S^0\}} (\langle S \mid T'(\mu) \mid S^0\rangle_0 h'(S^0,0) + \frac{\partial}{\partial\varepsilon}\langle S \mid T'(\mu) \mid S^0\rangle_{\varepsilon=0} W'(S^0)$$

where the dependence of $T$ on $\varepsilon$ has been made explicit. This equation describes the relaxation to the stationary state from a non stationary configuration. Compare this with [28] where the relaxation to the equilibrium state after a perturbation (i.e. magnetic field) was studied in a similar way.

the rescaled parameter set $\mu' = f(\mu)$. This implies that the renormalized single time distribution $P'(S,t')$ has the same functional form of the original distribution $P(\sigma,t)$. Since we are dealing with discrete evolution operators $T$, we define the time scaling factor $b^z$ as the average number of steps we apply the operator $T$ in order to obtain that $T'(\mu) = T(\mu')$ for the coarse grained system. It therefore depend upon the parameters set $\mu$. This condition defines the time recursion relation $g(\mu)$, from which it is possible to calculate the dynamical critical exponent $z$ (see Eq. 7).

## 3.2 Driving condition and recursion relations

The scheme discussed so far is a general formulation valid for each system which exhibits a stationary state, and its application presupposes the knowledge of the explicit form of the steady-state distribution $W$. For example, in equilibrium phenomena $W$ is given by the Gibbs distribution In that case it is possible to apply several methods such as cumulant expansions, exact or approximate decimation, to obtain the form of the recursion relations. For non-equilibrium dynamical system in general we do not know the form of the steady-state distribution. We will therefore develop an approximate method to evaluate the stationary distribution to be used in the calculation of the renormalized master equation.

The steady-state distribution can in general be split in two part

$$W(\sigma) = W^{(\mathrm{i})}(\sigma) + W^{(\mathrm{c})}(\sigma) \tag{19}$$

where $W^{(\mathrm{i})}(\sigma)$ and $W^{(\mathrm{c})}(\sigma)$ are, respectively, the incoherent and coherent part of the distribution. The incoherent part of the distribution has the property

$$\sum_{\{\sigma\}} \sigma_i \sigma_j W^{(\mathrm{i})}(\sigma) = \bar{\sigma}^2. \tag{20}$$

where $\bar{\sigma}$ is the single site average. Hence, it does not include correlations among variables and expresses a mean field approximation for the system. The coherent part $W^{(\mathrm{c})}(\sigma)$ can be subdivided in parts describing different kind of correlations: nearest-neighbors, next-nearest-neighbors etc. The incoherent part is a factorized distribution, that, for systems characterized by a q-state variables (see sect.2), has the form

$$W^{(\mathrm{i})}(\sigma) = \prod_i \langle \rho_{\sigma_i} \rangle \tag{21}$$

where $\langle \rho_\kappa \rangle$ is the average density of sites in the $\kappa$-state. In this way, we have approximated the probability of each configuration $\{\sigma_i\}$ as the product measure of the mean field probability to have a state $\sigma_i$ in each corresponding site. The incoherent part contribution to the renormalization equation is therefore particularly easy to obtain: for non-equilibrium steady-state we can use the stationarity condition, $\mathcal{S}_\mu(\{\langle \rho_\kappa \rangle\}) = 0$, to evaluate the densities $\langle \rho_\kappa \rangle$. By solving the stationary condition equation, the average densities of the $\kappa$-states for the coarse grained system are obtained as a function of the dynamical parameters $\mu$ at the corresponding iteration of the RG equations. By inserting this approximate distribution in Eq. (18), we get the following set of renormalization equations

$$\langle S \mid T(\mu') \mid S^0 \rangle = \frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S^0, \sigma^0) \mathcal{R}(S, \sigma) \langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle \prod_i \langle \rho_{\sigma_i^0} \rangle}{\sum_{\{\sigma^0\}} \mathcal{R}(S^0, \sigma^0) \prod_i \langle \rho_{\sigma_i^0} \rangle} \tag{22}$$

$$\mathcal{S}_\mu(\{\langle \rho_\kappa \rangle\}) = 0 \tag{23}$$

where the second equation denotes the dynamical steady state condition that allows to evaluate the approximate stationary distribution at each coarse graining scale. We call the Eq. (23) the *driving condition*, since it drives the RG equations acting as a feedback on the scale transformation.

Rewriting these equations in the form of recursion relations and adding the equation for the rescaling of time (see appendix A) we obtain [5]:

$$\mu' = f(\mu, \{\langle \rho_\kappa \rangle\}) \tag{24}$$

$$t' = t \; g(\mu, \{\langle \rho_\kappa \rangle\}) \tag{25}$$

$$\mathcal{S}_\mu(\{\langle \rho_\kappa \rangle\}) = 0 \qquad \Longrightarrow \qquad \langle \rho_\kappa \rangle = u_\kappa(\mu) \tag{26}$$

in which the driving condition appears explicitly. The above set of equations (or, equivalently, the Eq.s (22)-(23) synthesizes the *Dynamically Driven*

[5] Note that the $\mu$ recursion relation has to be read as a vectorial equation for the complete dynamical parameters set.

*Renormalization Group* at the lowest order. The fixed points $\mu^* = f(\mu^*)$ of the first equation govern the critical behavior of the system. The second equation gives the dynamical critical exponent $z$:

$$z = \frac{\log g(\mu^*)}{\log b} \tag{27}$$

The third equation, the driving condition, can be seen as a feedback mechanism between the dynamical and stationary properties of the system. In addition, from the fixed point solutions

$$\langle \rho_\kappa \rangle^* = u_\kappa(\mu^*) \tag{28}$$

the average stationary properties for the critical state are obtained.

In this form of the DDRG, we take into account only the uncorrelated part of the steady-state probability distribution. The results obtained are not trivial because correlations in the systems are considered in the dynamical renormalization of the operator $T$, that given a starting configuration traces all the possible paths leading to the renormalized final configuration. Moreover, geometrical correlations are treated by the operator $\mathcal{R}$ that maps the system by means of spanning conditions or majority rules. The renormalized uncorrelated part of the stationary distribution is evaluated from the stationary condition with renormalized parameters, thus providing an effective treatment of correlations.

In principle we can also refine the method by including higher order contributions to the unknown steady-state distribution. We have considered the simplest approximation for the driving condition, that takes into account only the single point densities. It is possible to consider also two points occupation probabilities, or even higher order clusters. One would then write generalized mean field equations [32] for the $n-$ point probability distributions and couple their solution to the renormalization of the dynamical operator. In other words, at every iteration of the scale transformation one should solve the generalized mean-field equations with *renormalized* couplings $\mu$. This is in the spirit of RSRG calculations where clusters of spins of higher order are introduced progressively in the calculations [26].

In the next section we present the explicit implementation of the method to a specific case, and we outline the results obtained for sandpile and forest-fire models.

# 4 DDRG applied to sandpile and forest-fire models

Many extended dissipative dynamical systems form structures with long range spatial and temporal correlations. The concept of self-organized criticality (SOC) has been invoked by Bak, Tang and Wiesenfeld [13] to describe the tendency of dynamically driven systems to evolve spontaneously toward a critical stationary state with no characteristic time or length scale, without the fine tuning of external parameters. An example of this behavior is provided by sandpile models: sand is added grain by grain on a $d$-dimensional lattice until unstable sand (too large local slope of the pile) slides off. In this way the pile reaches a steady-state, in which additional sand grains fall off the pile by avalanche events. The steady-state is critical since avalanches of any size are observed. This class of models can be used to describe a generic avalanche phenomenon, interpreting the sand as energy, mechanical stress or heat memory.

The common characteristic of SOC systems is that the dynamical process creates spontaneously a critical stationary state, i.e., a non-equilibrium critical steady-state. We can therefore apply the DDRG in order to study their critical behavior. We studied the critical height sandpile automaton [13] and the forest-fire model [19, 20]. These two models have been intensely studied numerically and can be considered as mile-stones in the field of SOC phenomena. In what follows we will show that the DDRG allows us to calculate analytically the critical exponents and to clarify the SOC nature of both models: we are able to study the fixed point and to identify the control and the order parameter of the models.

## 4.1 Forest-fire model

The Forest-Fire model (FFM) has been introduced by Bak et al.[19] as a an example of SOC, and has been then modified by Drossel and Schwabl [20].

The model is defined on a lattice in which each site can be empty ($\sigma_i = 0$), occupied by a green tree ($\sigma_i = 1$) or by a burning tree ($\sigma_i = 2$) (see Fig 1). At each time step the lattice is updated as follows:

i) a burning tree becomes an empty site;

ii) a green tree becomes a burning tree if at least one of its neighbors is burning;

iii) a tree can grow at an empty site with probability $p$;

iv) a tree without burning nearest neighbors becomes a burning tree with probability $f$.

The system was supposed to exhibit SOC under the hypothesis of a double separation of time scales: trees grow fast compared with the occurrence of lightnings and forest clusters burn down much faster than trees grow. This request is expressed by the double limit $\theta \equiv f/p \to 0$ and $p \to 0$.

The critical state is characterized by a power law distribution $P(s) = s^{-\tau}$ of the forest clusters of $s$ sites (avalanches in the SOC terminology) and the average cluster radius (the correlation length) scales as $R \sim \theta^{-\nu}$.

In the past few years, a great amount of work has been done in order to describe the critical state of the Forest-Fire model and to calculate the critical exponents. Numerical simulations [34, 35] show that in the time scale separation regime the model is close to a critical point with the avalanche critical exponent $\tau$ given by $\tau \simeq 1$ in $d = 1$ and $\tau \simeq 1.15$ in $d = 2$. For the exponent $\nu$, describing the divergence of the average cluster radius as $\theta \to 0$, it has been found $\nu \simeq 1$ in $d = 1$ and $\nu \simeq 0.58$ in $d = 2$. The one dimensional result has been recovered exactly in [36]. Simulations were performed also in higher dimensions [35]. In the limit $d \to 6$ the exponents approach those of mean field percolation.

To apply the DDRG to this problem we first have to chose a form for the coarse graining operator. For the sake of simplicity let us consider in full detail the one-dimensional case. We use a cell-to-site transformation with scale factor $b = 2$. In this case the operator $\mathcal{R}$ can be written in the following way:

$$\mathcal{R}(S, \sigma) = \prod_J \mathcal{R}(S_J, \{\sigma_i\}_J) \tag{29}$$

where each term is acting on a specific cell $J$ and $\{\sigma_i\}_J$ denotes the configurations of spins belonging to that cell. Therefore, given a two sites cell, the operator $\mathcal{R}$ renormalizes it in a coarse grained site following only "inside the cell" rules. The rules defining $\mathcal{R}$ are as follows. A two sites cell is renormalized as a tree site if it is spanned from left to right by a connected path of green sites. Accordingly, a cell is empty if it is not spanned by a connected path of green sites. Finally we consider a cell as burning if it contains at least one burning site. In this last case the spanning condition that ensures connectivity properties is not necessary because fire spreads automatically to nearest neighbor sites. The above renormalization prescription, i.e. the operator $\mathcal{R}$, is summarized in Fig. 2. We denote with an index $\alpha$ each two sites configuration $\{\sigma_i\}_J$ so that $\sum_{\{\sigma_i\}_J} \to \sum_\alpha$. The dynamical rules of the FFM are local, therefore we can define matrix elements reduced to a single site[6] for the dynamical operator $T$:

$$\langle 0 \mid T \mid 2\rangle = 1 \tag{30}$$

$$\langle 0 \mid T \mid 0\rangle = 1 - p \quad ; \quad \langle 1 \mid T \mid 0\rangle = p \tag{31}$$

$$\langle 1 \mid T \mid 1\rangle = 1 - f \quad ; \quad \langle 2 \mid T \mid 1\rangle = f \tag{32}$$

In addition to the above rules, fire spreads between nearest neighbor sites. We adopt a finite lattice truncation of the renormalization equations (22,23) on the two sites cells subspace defined by the operator $\mathcal{R}$, and we obtain

$$\langle S_i \mid T' \mid S_i^0\rangle = \frac{\sum_\alpha \sum_{\alpha'} \langle \alpha' \mid T^{b^z} \mid \alpha\rangle W_\alpha}{\sum_\alpha W_\alpha} \tag{33}$$

where $\mid \alpha\rangle$ and $\mid \alpha'\rangle$ are the two sites cells states[7] which renormalize respectively in $\mid S_i^0\rangle$ and $\mid S_i\rangle$. We keep the subscript $i$ for the latter states because they are referring to a single coarse grained site and not to a system's configuration. With $W_\alpha$ we denote the stationary statistical weight of each $\alpha$ configuration.

Let us now evaluate explicitly the above equations. Because we are interested in the critical behavior for $f \ll 1$ and $p \ll 1$ we can write non trivial RG equations keeping only terms up to the first order in $p$ and $f$. In addition, in order to define consistently the recursion relations, the operator $T'$ must preserve its form at each scale; i.e. no proliferations are allowed. This implies that

$$\langle S_i = 0 \mid T' \mid S_i^0 = 2\rangle = 1 + \mathcal{O}(p^2, f^2, pf) \tag{34}$$

[6] $\mid 0 >$, $\mid 1 >$ and $\mid 2 >$ are states in which the site $i$ is in the corresponding $\sigma_i$ state irrespective of the remaining of the system.

[7] For instance $\mid \alpha = 1\rangle = \mid 1, 1\rangle$ (see fig.2).

where the higher order terms will be neglected in the recursion relations. This implies that new parameters are not introduced in the description of the system. Thus, to avoid proliferations in the burning event, we have to define a dynamical operator $T^{b^z}$ that leaves invariant Eq. (34). It easy to check that this is the case if $T^{b^z} = T^2$; i.e. $z = 1$. In two time steps a burning cell evolves always in an empty one (Fig. 3a) if we neglect second order contributions in $p$ and $f$. In Fig. 3b we show a possible proliferation, which however has a weight $p^2$, and can therefore be neglected. Avoiding proliferations in the burning event defines unambiguosly the time scaling factor: the relevant time scale in the system is that of the burning process, as was already pointed out on the basis of numerical simulations [20].

The renormalization of the lightning probability in this framework is straightforward. We have only one starting configuration, i.e. $\mid \alpha = 1\rangle$ and the recursion relation is

$$f' = \langle S_i = 2 \mid T' \mid S_i^0 = 1\rangle = \sum_{\alpha'=4}^{6} \langle \alpha' \mid T^2 \mid \alpha = 1\rangle = 4f + \mathcal{O}(p^2, f^2, pf) \quad (35)$$

In the same way we obtain the expression for the renormalized growth probability $p$ as

$$\langle S_i = 1 \mid T' \mid S_i^0 = 0\rangle = \frac{\sum_{\alpha=2}^{3} \langle \alpha' = 1 \mid T^2 \mid \alpha\rangle W_\alpha}{\sum_{\alpha=2}^{3} W_\alpha}, \quad (36)$$

from which follows

$$p' = 2p\frac{W_3}{W_2 + W_3} + \mathcal{O}(p^2, f^2, pf). \quad (37)$$

A process that contributes to the above equations is shown in Fig. 4a. Finally, we have also to treat the normalization of the operator $T'$. In Fig. 4b we show a proliferation given by a process in which with probability $f$ an empty cell becomes a burning cell at the coarse grained level. In the previous equations, this last process is present only because of the normalization condition:

$$\langle S_i = 1 \mid T' \mid S_i^0 = 0\rangle = 1 - 2p\frac{W_3}{W_2 + W_3} - f\frac{W_3}{W_2 + W_3} + \mathcal{O}(p^2, f^2, pf) \quad (38)$$

In order to avoid this proliferation we have to impose a supplementary condition. If we restrict our analysis to the region in which $f \ll p$, we can neglect terms linear in $f$ where terms linear in $p$ are present. This corresponds to truncate Eq.(38) by keeping only the term linear in $p$, and thus eliminating the proliferation. The inclusion of this proliferation would describe the behavior of the model in the limit $f \simeq p$ which, from numerical simulations, is expected to be different. It is worth to remark that the above approximation corresponds to renormalize in a separate way the tree growth parameter $p$ and the lightning parameter $f$, assuming that they do not affect each other since they act on very different time scales.

The steady state probability distribution $W_\alpha$ is approximate following the DDRG general scheme in the lowest order

$$W_\alpha = \prod_{i=1}^{2} \langle \rho_{\sigma_i} \rangle = n \langle \rho_{\sigma_1} \rangle \langle \rho_{\sigma_2} \rangle \tag{39}$$

where $n$ takes into account the multiplicity due to symmetries of each configuration.

We can obtain the densities in the steady state from the following dynamical mean-field equations:

$$\langle \rho_0(t+1) \rangle = (1-p)\langle \rho_0(t) \rangle + \langle \rho_2(t) \rangle \tag{40}$$

$$\langle \rho_1(t+1) \rangle = \langle \rho_0(t) \rangle p + (1-t)\langle \rho_1(t) \rangle (1 - \langle \rho_2(t) \rangle)^{2d-1} \tag{41}$$

$$\langle \rho_2(t+1) \rangle = \langle \rho_1(t) \rangle (1 - (1-f)(1 - \langle \rho_2(t) \rangle)^{2d-1}) \tag{42}$$

where $d$ is the spatial dimension (see Ref.[35] for their derivation). The long time limit ($t \to \infty$) solution of the above equations provides the driving condition; i.e. the average steady state densities.

Collecting all these equations we obtain the DDRG recursion relations for the Forest Fire model, that in one dimension read as follows:

$$\begin{cases} p' = 2p \frac{\langle \rho_1 \rangle}{\langle \rho_1 \rangle + \frac{1}{2}\langle \rho_0 \rangle}; \\ \theta' = 2\theta \cdot \frac{\langle \rho_1 \rangle + \frac{1}{2}\langle \rho_0 \rangle}{\langle \rho_1 \rangle}; \\ g(\theta, p, \rho) = 2 \\ \langle \rho_0 \rangle - (1 - \langle \rho_1 \rangle) a/p = 0; \\ \langle \rho_1 \rangle - \frac{a}{\theta p + 4 \cdot a} - a \cdot \langle \rho_1 \rangle = 0; \\ \langle \rho_2 \rangle - (1 - \langle \rho_1 \rangle) a = 0. \end{cases} \tag{43}$$

where we defined $a = p/(1+p)$. We express the recursion relations in terms of the parameter $\theta$, in order to compare with numerical simulation. It is important to emphasize again that the recursion equations (43) are valid only in the double time scale separation $f \ll p \ll 1$ which defines the range of validity for our approximations. This limit is the one for which the FFM shows non trivial critical behavior.

Eqs. (43) have a fixed point for $\theta = 0$, $p = 0$, $\rho_1 = 1$. The fixed point is repulsive along the $\theta$ direction, thus defining the critical exponent $\nu$ in term of the largest eigenvalue $\lambda$ of the linearized renormalization equation:

$$\nu = \frac{\log 2}{\log(\lambda)} = 1 \tag{44}$$

From simple scaling relations it is possible to obtain also the other critical exponents which are summarized in table I. In this respect it is interesting to note that our method yields in the one dimensional case the exact results of the rigorous treatment of ref.[36]. This is due to the relative simplicity of the one dimensional case, where the approximations involved - i.e. spanning conditions or proliferations - are irrelevant.

In $d = 2$ the calculation of the RG equations proceeds along the lines shown above [18], but are algebraically more complex. In table I the results obtained in $d = 2$ for $b = 2$ and $b = 3$ are compared with numerical simulations. The existence of a relevant scaling field and the general structure of the flow diagram is stable with respect to different approximation schemes, and more refined calculations lead systematically to an improvement in the numerical values of the results. The FFM is critical along the line $\theta = 0$ of the phase space, so that $\theta$ is equivalent to the reduced temperature in thermal phase transitions. In other words $\theta$ is the control parameter of the model, and the critical state is reached only by a fine tuning of $\theta$ to its critical value. The control parameter $\theta$ is the ratio between two very different time scales, controlled by $f$ and $p$, with a critical value fixed to zero. In this situation, however, the existence of a time scale separation makes the system very close to the critical point without an apparent fine tuning of internal parameters. Strictly speaking however, the system is critical just in correspondence of the critical value of the control parameter.

## 4.2 Sandpile models

Sandpile models are cellular automata [13, 37] defined in a $d-$ dimensional lattice. A variable $E(i)$, that we denote by energy, is associated with each lattice site $i$. At each time step an input energy $\delta E$ is added to a randomly chosen site. When the energy on a site reaches a threshold value $E_c$ the site relaxes transferring energy to the neighboring sites:

$$E(i) \to E(i) - \sum_e \Delta E(e) \tag{45}$$

$$E(i+e) \to E(i+e) + \Delta E(e) \tag{46}$$

where $e$ represent the unit vectors on the lattice. A typical choice for the parameters is , for example, $E_c = 4$ and $\Delta E(e) = \delta E = 1$, but other possibilities have also been considered. The relaxation of the first site can induce a series of relaxations generating an avalanche. Note that the energy is added to the system only when the configuration is stable (i.e. all the sites are below the threshold). The boundary conditions are usually chosen to be open so that energy can leave the system. In these conditions the system organizes itself in to a stationary state characterized by avalanches of all length scales. In particular the distribution for avalanches sizes $s$ decays as a power law $P(s) \sim s^{-\tau}$, and the linear size of the avalanche scales with time $r \sim t^z$.

This model has been extensively studied in the past by means of numerical simulations [38, 39, 40] and several exact results have been derived for Abelian sandpiles [41].

In the steady state each configuration of the system can be described by the energy probability distribution $W(\{E_i\})$. In the zero order approximation we consider the incoherent part of the distribution:

$$W^{(\mathrm{i})}(\{E_i\}) = \prod_i w(E_i) \tag{47}$$

where $w(E)$ is the single site energy distribution. The average energy of a site evolves according to the following equation, written in continuum notation:

$$\frac{dE(t)}{dt} = \delta E_{in} - \delta E_{out} \tag{48}$$

where $\delta E_{in}$ is the average energy entering into the site either because of a relaxation in a neighboring site or because of the external perturbations, and $\delta E_{out}$ is the average energy dissipated by the site. At lower scale $\delta E_{out} = \rho \sum_e \Delta E(e)$, where we have defined

$$\rho = \int_{E_c - \delta E}^{E_c} w(E) dE \tag{49}$$

as the probability that a site relaxes.

This equation suggest a simple way to describe the steady state of the model. At any scale, we can divide the sites in *critical* ($\sigma_i = 1$) and *stable* ($\sigma_i = 0$). Stable sites do not relax when energy is added to them. On the other hand critical sites relax when they receive an energy grain $\delta E_{in}$. In this formalism $\rho$ represent the density of critical sites. For convenience we will also define unstable sites, as those that are relaxing, even though they are not present in the static configurations of the system (see Fig. 1). These definitions can be extended to a generic scale $b$. For instance, a cell at scale $b$ is considered critical if the addition of energy $\delta E_{in}(b)$ induces a relaxation of the size of the cell (i.e. the avalanche spans the cell).

To construct the DDRG recursion relations we have to describe the relaxation at a coarse grained scale. We first note that the only non trivial matrix element of the time evolution operator is the one describing the evolution of a cell from critical to stable. This process occurs when a critical cell becomes unstable due to the addition of energy. The inverse process, a stable cell becoming critical, is simply due to the accumulation of energy and is not characterized by critical exponents which, on the other hand, describe the avalanche propagation.

In a relaxation at the minimal scale energy is distributed equally in the four directions. This is no longer the case at a coarse grained level where different possibilities arise: the energy in principle can be distributed to one, two, three or four neighbors (Fig. 5). These events occur with probabilities

$$\vec{P} = (p_1, p_2, p_3, p_4) \tag{50}$$

In terms of the matrix element $\langle 0|T|1\rangle$ the vector $\vec{P}$ represents a conditional probability

$$p_n = \frac{\langle 0|T|1\rangle_n}{\langle 0|T|1\rangle} \tag{51}$$

where $\langle 0|T|1\rangle_n$ is the probability that a critical site becomes empty and transfers energy to $n$ neighbors. In this way we have obtained the set of parameters that describes the dynamics. Of course the choice of the parameters space is not uniquely determined, one encounters proliferation problems typical of real space RG methods.

The renormalized matrix element is then obtained by considering all the processes that span the cell and transfer energy outside. This rule implicitly defines the effective dynamical operator $T^{b^z}$. The static projector $\mathcal{R}$, on the other hand, select those cells that can be considered stable, critical or relaxing sites at the coarser scale, the renormalized parameters being:

$$p'_n = \frac{\sum_{\{\sigma^0\}}\sum_{\{\sigma\}}\mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma)\langle\sigma \mid T^{b^z} \mid \sigma^0\rangle_n W(\sigma^0)}{\sum_{\{\sigma^0\}}\sum_{\{\sigma\}}\mathcal{R}(S^0,\sigma^0)\mathcal{R}(S,\sigma)\langle\sigma \mid T^{b^z} \mid \sigma^0\rangle W(\sigma^0)} \tag{52}$$

Here, $\langle\sigma \mid T^{b^z} \mid \sigma^0\rangle_n$, denotes the relaxation processes affecting $n$ coarse grained cells. We proceed in defining a renormalization procedure for the dynamics by considering a finite truncation on four-sites cells. This corresponds to a cell-to-site transformation on a square lattice, in which each cells at the coarser scale is formed by four sub-cells at the finer scale: the length scaling factor is $b = 2$. Every cell at the coarser scale can be characterized by an index $\alpha$ that indicates the number of critical sub-cells, and we have that $\sum_{\{\sigma_i\}} \to \sum_\alpha$. The approximated stationary distribution (Eq. (21)) for each of these configurations is given by:

$$W_\alpha(\langle\rho\rangle) = n_\alpha \prod_{i=1}^{4}\langle\rho_{\sigma_i}\rangle = n_\alpha\langle\rho\rangle^\alpha(1-\langle\rho\rangle)^{4-\alpha} \tag{53}$$

where $n_\alpha$ is a factor due to the multiplicity of each configuration. The dynamical renormalization can then be rewritten in a simpler form. For each starting cell we can define the set $\omega_n$ of paths at the finer scale that renormalize in a $p'_n$ process at the coarser scale (i.e. they transfer energy to $n$ larger scale cells), and the respective statistical weight $f_{\omega_n}$ of this set, which is function of the parameters $p_{n'}$ at the finer scale. We can write this in synthetic form as

$$\sum_{\alpha'}\langle\alpha' \mid T^{b^z}(p_{n'}) \mid \alpha\rangle_n = \sum_{\{\omega_n\}_\alpha} f_{\omega_n}(\{p_{n'}\}); \qquad n' = 1\ldots 4 \tag{54}$$

where $\mid\alpha\rangle$, $\mid\alpha'\rangle$ denotes the four sites configurations which renormalize in $\mid S_i^0 = 1\rangle$ and $\mid S_i = 0\rangle$, respectively. By using this scheme and replacing sums over configurations with sums over the index $\alpha$, the recursion relations become

$$p'_n = \frac{1}{\mathcal{N}}\sum_{\alpha=2}^{4} W_\alpha(\langle\rho\rangle)\sum_{\{\omega_n\}_\alpha} f_{\omega_n}(\{p_{n'}\}) \tag{55}$$

where the denominator of eq.(52) is adsorbed in the normalization factor $\mathcal{N}$.

The time scaling factor, by applying considerations discussed in the appendix, is given by

$$g(p_n) = \frac{1}{\mathcal{N}} \sum_{\alpha,n} W_\alpha(\langle\rho\rangle) \sum_{\{\omega_n\}_\alpha} f_{\omega_n}(\{p_{n'}\}) t_{\omega_n} \tag{56}$$

where $t_{\omega_n}$ denotes the number of non contemporary relaxation events in each path $\omega_n$.

The driving condition (eq.(23)) is obtained from eq.(48) by imposing stationarity. This implies that the stationary state is characterized by the balance between the energy that goes in and the energy that goes out of the system. We assume that energy is transferred in "quanta" $\delta E = \delta E_{in}$ in each direction and we obtain on average

$$\delta E = \langle\rho\rangle \delta E \sum_n n p_n \tag{57}$$

which implies

$$\langle\rho\rangle = \frac{1}{\sum_n n p_n} \tag{58}$$

This relation gives the average density of critical sites in the steady-state, allowing us to evaluate the approximate stationary distribution at each scale. Therefore, eq.s(55),(56) and (58) are the complete set of the DDRG recursion relations.

The explicit evaluation of the recursion relations depends upon the chosen spanning condition. In ref.[17], the scheme considers only connected paths that span the cell from left to right or top to bottom. This spanning rule implies that only paths extending over the size of resulting length scale contribute to the renormalized dynamics, and it ensures the connectivity properties of the avalanche in the renormalization procedure.

An example of such a path is shown in Fig. 6. In this case $\alpha = 2$ and the path shown refers to the probability that the unstable sub-cell relaxes towards the other critical sub-cell [Fig.6(b)]. This occurs with probability $(1/4)p_1$. At this point we consider the probability that the next relaxation event at the fine scale involves two neighboring sites, one inside and one outside the original cell of size $b = 2$ [Fig.6(c)].This occurs with probability $(2/3)p_2$. This series of relaxation processes contributes to the weight $f_{\omega_1}$

that characterizes the relaxation processes at the coarse grained scale. By summing over all the paths that lead to a $p'_1$ process one obtains for $\alpha = 2$

$$\begin{array}{l} f_{\omega_1} = (\frac{1}{4}p_1 + \frac{1}{6}p_2)(\frac{1}{2}p_1 + \frac{2}{3}p_2\frac{1}{2}p_3) + \\ +(\frac{1}{6}p_2 + \frac{1}{4}p_3)(\frac{1}{2}p_1 + \frac{1}{6}p_2) + (\frac{1}{6}p_2 + \frac{1}{4}p_3)(\frac{1}{6}p_2 + \frac{1}{4}p_3)(\frac{3}{4}p_1 + \frac{1}{2}p_2\frac{1}{4}p_3). \end{array} \tag{59}$$

In a similar way one can also write the expression for $f_{\omega_2}$, $f_{\omega_3}$ and $f_{\omega_4}$. Proliferation effects due to multiple relaxations of the same site and sites becoming critical during the dynamical process are not considered in this scheme. The complete polynomials for $f_{\omega_n}$ weights involve more than two hundred terms and are reported in ref.[17].

The flow diagram and the relative fixed point in the parameter space can then be studied. We find an attractive fixed point: the parameters evolves spontaneously towards their critical value. From the fixed point we can compute the critical exponents. For a square cell of size $b = 2$ we obtain $\tau = 1.25$ and $z = 1.17$, in good agreement with computer simulations (see table II). The same method has also been applied to dissipative sandpile models [17] and to directed sandpile models [42]

Recently, our approach has been improved in ref.[43]. The expressions $f_{\omega_n}$ are linked to a branching mechanism that allows their calculation through a generating function. Using this method, is possible to include more proliferations to the set of relaxation processes considered. The results obtained with this improved scheme allows an excellent qualitative and quantitative description of critical height sandpile model.

It is worth to remark that in this scheme, as is usually done in computer simulations, we are implicitly assuming a *slow driving* condition for the model. In fact, in the evaluation of the RG equations the external drive does not interfere with the relaxation process: dynamical processes during which sites become critical are not considered. In order to overcome this approximation the energy flow on its turn should be renormalized and the addition of energy (possibility of new relaxation events) during relaxation processes be allowed. Enlarging the phase space, a relevant parameter appears, e.g. the driving rate. The driving rate is the incoming current of energy per unit time with respect to the total average energy in the system. Also in this case, as in the FFM, the system would be strictly critical just in correspondence of infinitesimal driving rate. From this point of view the only difference between Sandpile models and Forest-Fire models is that the latter can not be studied in a subspace with no relevant parameters without destroying the model itself. The meaning of the self-organization is then related to the widespread existence of systems with very different time scales and not to the absence of relevant control parameters as often reported in literature.

# 5 DDRG and equilibrium critical phenomena

The DDRG represents a general method to approach non-equilibrium critical systems with a stationary state and it allows also to study equilibrium models at the critical point. In this last case the stationary state is characterized by probability densities written in terms of the Hamiltonian of the system by means of the Gibbs distributions. This kind of systems have been extensively studied and well established theoretical tools are available to approach them. The prototype of such systems is the Ising model. On each site $i$ of some finite dimensional lattice we place a random variable $\sigma_i$ taking the values $\pm 1$. The Hamiltonian is

$$H(\sigma) = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \tag{60}$$

where the sum runs over all the nearest neighbors pairs. The correspondent Gibbs measure is

$$W(\sigma) = Z^{-1} \exp\{-\beta H(\sigma)\} \tag{61}$$

where $Z$ is a normalization factor (the partition function).

In order to apply the DDRG to equilibrium critical phenomena one has to treat these systems as dynamical systems with a well-defined stationary state. Our discussion will refer to the Ising model with *heat bath* dynamics but the arguments can be generalized to different cases. In a system described by a state $\{\sigma_i\}$, the conditional probability to flip a spin in a site $i$ is given by

$$\langle \sigma_i | T(\beta) | \sigma_i^0 \rangle = \frac{1}{1 + e^{\sigma_i \beta [J \sum_{nn} \sigma_j^0]}} \tag{62}$$

where $nn$ indicates the nearest-neighbors of the site $i$ and $\beta$ is the inverse of the temperature. The configuration $\{\sigma_i\}$ is obtained from $\{\sigma_i^0\}$ by flipping the spin $i$. The heat-bath algorithm updates $\sigma_i^0$ by choosing a new spin value $\sigma_i$, independently of the old value of $\sigma_i^0$, from the conditional probability given by Eq. (62). All the others spins remain unchanged. If $\{\sigma_i\}$ and $\{\sigma_i^0\}$

are two arbitrary configurations of the system and $\langle\sigma|T(\beta)|\sigma^0\rangle$, $\langle\sigma^0|T(\beta)|\sigma\rangle$ are the 1-step transition probabilities between the two configuration, the following relation, known as detailed balance, is satisfied by the heat-bath dynamics

$$W(\sigma)\langle\sigma|T(\beta)|\sigma^0\rangle = W(\sigma^0)\langle\sigma^0|T(\beta)|\sigma\rangle \tag{63}$$

where $W(\sigma)$ and $W(\sigma^0)$ are the stationary (equilibrium) distributions for the states $\{\sigma_i\}$ and $\{\sigma_i^0\}$.

From the detailed balance we can deduce a stationarity condition for the dynamics which provides the steady-state distribution to be used in the DDRG. For the actual implementation of the RG procedure one can use standard techniques for the definition of the operator $\mathcal{R}$, e.g. majority rules, spanning conditions etc., and for the dynamical renormalization of the parameters $\mu$. The work in this direction is still in progress and it will provide a test for the flexibility of our approach. In order to test the accuracy of the method we could compare the results for the probability densities obtained in our framework with those obtained renormalizing directly the Gibbs distributions with usual RG schemes.

# 6 Conclusions.

In this paper we have presented a new renormalization scheme especially suited for systems with non-equilibrium critical steady-state. The essential idea of the method is the use of an approximate stationary probability distribution for the configurations of the system. This distribution is evaluated through a driving condition which identifies the single time averages in the steady-state. This approximate distribution is used as a weight in the renormalization of the master equation which takes into account correlations due to the dynamical evolution. These correlation are thus considered in the approximate probability distribution of the renormalized system, which is calculated by the driving condition with renormalized dynamical parameters. The dynamical renormalization of the master equation is based on real space RG schemes which, in spite of their low systematicity, are very simple and intuitive. In addition, these schemes leave room for "higher order" techniques to improve the accuracy of the results.

The DDRG can be used to study systems with stationary critical state, but is particularly useful in non-equilibrium systems for which it is impossible to obtain the stationary probability distribution in the configuration space. In fact, the DDRG appears to be extremely powerful in the study of SOC systems. These systems show a non-equilibrium steady-state very close to the critical point for a wide range of internal parameters.

The application to forest-fire and sandpile models shows that the general results obtained are not affected by the approximations involved in our scheme, even though a more refined treatment of the calculation scheme, i.e. by introducing more proliferations, leads to an improvement of the values obtained for the critical exponents. For SOC models the DDRG can be considered as a general theoretical framework that provides the analytical tools for a qualitative and quantitative study of the critical stationary state.

We considered as well the application to usual equilibrium phenomena, in the perspective of a better understanding of the method and its application to other systems. In particular the DDRG could be very effective to study the critical properties of driven diffusive systems [9, 10], which to our knowledge have never been approached by real space RG methods. Work in this direction is currently in progress.

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## Appendix A

The explicit evaluation of the time scaling factor and the corresponding recursion relation is a complex task, because there is no standard recipe for this in real space RG schemes. Here we show a general strategy, which depends on the specific model for its actual implementation.

The effective operator $T^{b^z}$ contains all the dynamical processes that contribute to the definition of a meaningful renormalized dynamics. We define the following transformation

$$\langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle = \sum_N \mathcal{D}_N \langle \sigma \mid T^N(\mu) \mid \sigma^0 \rangle \tag{A. 1}$$

where $\mathcal{D}_N$ is the renormalization operator for the dynamical evolution of the system: it is a projection operator that samples only the paths of $N$ time steps which have to be considered in the definition of the effective operator $T^{b^z}$. To clarify this point, let us consider for a moment a spin flip dynamics in a Ising-like system. The matrix element $T(\mu)$ is non zero only for those configurations that are related by a single spin flip. In order to preserve the same form for $T$ at a coarse grained scale, we have to impose that the renormalized time evolution operator connects only configurations that differ by a single coarse grained spin flip. Flipping a coarse grained spin corresponds to the subsequent flipping of different spins in the original system. The number of flipping necessary to flip a macroscopic spin is not uniquely defined, but depend on the configuration, both at the coarse grained and fine grained level, and on the particular dynamical path chosen. The effective operator $T^{b^z}$ is then a convolution of different $N$-steps operators.

The operator $\mathcal{D}_N$ is chosen on the basis of physical considerations: spanning conditions etc. In addition, $\mathcal{D}_N$ should satisfy some general properties in order to preserve the symmetry or the internal space of the dynamical variables. For instance, we have to ensure the normalization of the effective dynamical operator by the property

$$\sum_{\{\sigma\}} \sum_N \mathcal{D}_N \langle \sigma \mid T^N(\mu) \mid \sigma^0 \rangle = 1 \tag{A. 2}$$

Moreover, $\mathcal{D}_N$ must be consistent with the definition of the renormalization operator $\mathcal{R}$: it should describe dynamical processes among renormalized variables of the same type of those given by the operator $\mathcal{R}$. Finally, $\mathcal{D}_N$ has to preserve the form of the dynamical operator $T$ at each scale. This condition imposes that the time scaling is consistent with the length scaling used in $\mathcal{R}$. In this way it is possible to map the renormalized system in the old one with renormalized variables.

As previously mentioned the operator $\mathcal{D}_N$ can assume a very simple form:

$$\mathcal{D}_N = \delta_{N,N'} \tag{A. 3}$$

where $N' = b^z$. In general, however, more complicated expressions are encountered (see ref. [17, 18]), since $\mathcal{D}_N$ depends on the specific dynamics. We have defined the effective evolution operator so that $T'(\mu) = T(\mu')$ for the renormalized system. On the other hand, the operator $T^{b^z}$ is in general the convolution of the discrete time step operator $T^N$ projected by the renormalization operator $\mathcal{D}_N$. Thus, we can write it as a sum of terms that represent the statistical weight for the evolution paths of $N$ time steps:

$$\langle \sigma \mid T^{b^z}(\mu) \mid \sigma^0 \rangle = \sum_N \gamma^N_{\sigma,\sigma^0}(\mu) \tag{A. 4}$$

These terms, which are obtained by the specific definition of the operator $\mathcal{D}_N$, can be used to calculate the time scaling factor $b^z = g(\mu)$ as an average over the renormalized dynamical processes. In fact, we can interpret the right sum in Eq.(A. 4) as an integral over dynamical evolution paths of different time duration; the time scaling factor being an average over these paths whose statistical weights are the terms $\gamma^N_{\sigma,\sigma^0}(\mu)$. For each case we have to find the process which defines the relevant time scale of the phenomenon. The time scaling factor will be the average over contributing paths to the renormalization of this process.

To illustrate how the above procedure works in practice, we shall discuss explicitly the FFM and sandpile examples. For FFM we are in the simple case of Eq.(A. 3). In fact the relevant process, i.e. the burning process, determines $\mathcal{D}_N = \delta_{N,2}$. therefore only terms $\gamma^2_{\sigma,\sigma^0}(p,f) = \langle \sigma \mid T^2(p,f) \mid \sigma^0 \rangle$ are allowed, so we simply have $b^z = g(p,f) = 2$.

For the sandpile automata the calculation is rather laborious. In the slow driving regime, the only relevant time scaling length is given by the single relaxation event, for which we have at each scale

$$\langle S_i = 0 \mid T' \mid S_i^0 = 1 \rangle = 1 \tag{A. 5}$$

The operator $\mathcal{D}_N$ selects those spanning paths that lead to a relaxation process at the coarser scale. By using the cell-to site transformation defined in

Sec.4.2, it is easy to show

$$\sum_{\{\sigma\}} \mathcal{R}(S_i^0 = 1, \sigma^0)\mathcal{R}(S_i = 0, \sigma) \sum_N \gamma^N_{\sigma,\sigma^0}(p_{n'}) = \sum_n \sum_{\{\omega_n\}_\alpha} f_{\omega_n}(\{p_{n'}\}) \quad \text{(A. 6)}$$

where the right term has been defined as the sum of the statistical weights of paths that contribute to the renormalized relaxation event. The sum over $n$ denotes we are considering relaxation processes without distinguishing the number of affected nearest neighbors. Finally we have to write the equation that gives the time scaling factor from the total average over contributing processes to the renormalized matrix element $\langle 0 \mid T' \mid 1\rangle$ :

$$g(\mu) = \frac{\sum_{\{\sigma^0\}} \sum_{\{\sigma\}} \mathcal{R}(S_i^0 = 1, \sigma^0)\mathcal{R}(S_i = 0, \sigma) \sum_N N\gamma^N_{\sigma,\sigma^0}(p_n) W(\sigma^0)}{\sum_{\{\sigma^0\}} \mathcal{R}(S_i^0 = 1, \sigma^0) W(\sigma^0)}, \quad \text{(A. 7)}$$

By inserting Eq.(A. 6) in the above relation and defining $t_{\omega_n}$ as the time duration of each path $\omega_n$ we finally obtain

$$g(p_n) = \frac{1}{\mathcal{N}} \sum_\alpha W_\alpha(\langle\rho\rangle) \sum_n \sum_{\{\omega_n\}_\alpha} f_{\omega_n}(\{p_{n'}\}) t_{\omega_n} \quad \text{(A. 8)}$$

where we used the DDRG scheme to explicitly get the stationary weights and $\mathcal{N}$ is an opportune normalization factor.

The above relations will provide the consistent rescaling of time by imposing that $b^z = g(p_n^*)$ from which it is possible to calculate the dynamical critical exponent. This also shows that in general the factor $g(\mu)$ is a function of the dynamical parameters.

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| $d = 1$ | $\nu$ | $z$ | $\tau$ |
|---|---|---|---|
| RG | 1.0 | 1.0 | 1.0 |
| Exact results$^*$ | 1.0 | 1.0 | 1.0 |
| $d = 2$ | $\nu$ | $z$ | $\tau$ |
| RG 2 x 2 | 0.73 | 1.11 | 1.19 |
| RG 3 x 3 | 0.65 | 1.02 | 1.17 |
| Numerical results$^+$ | 0.58 | 1.04 | 1.15 |

Table 1: In this table we summarize our results for the Forest Fire Model critical exponents obtained with different approximation schemes. For comparison we report also the exact or numerical results. $^*$: Exact results from [36]. $^+$: Numerical results from [34].

| | $\tau$ | $z$ |
|---|---|---|
| RG scheme | 1.25 | 1.17 |
| BTW simulations | 1.22 | 1.21 |

Table 2: In this table the values of the critical exponents calculated with our RG scheme for the sandpile model are summarized. For comparison we also list the values obtained from large scale simulations on the BTW model [38].

TABLES

FIGURES

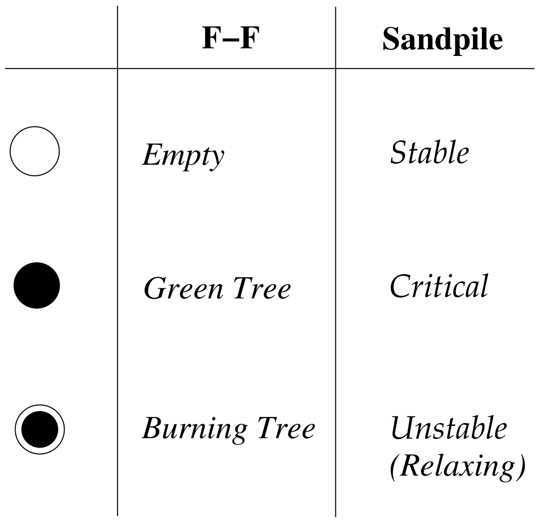


Figure 1: The sites classification in the Forest Fire and in the sandpile model.

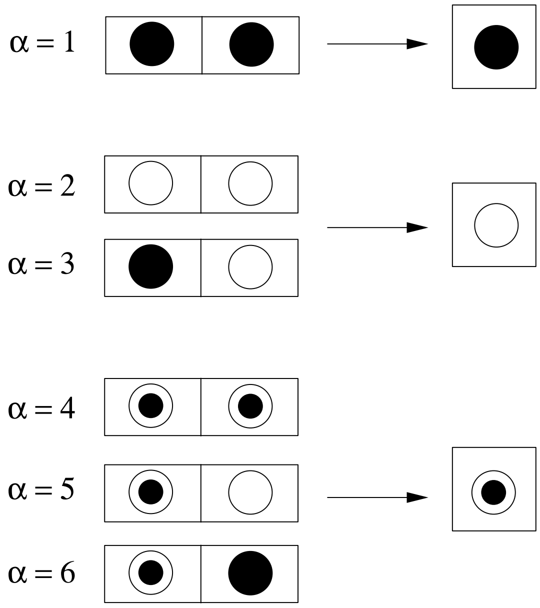


Figure 2: The rules defining the renormalization operator $\mathcal{R}$ for the one dimensional Forest Fire model.

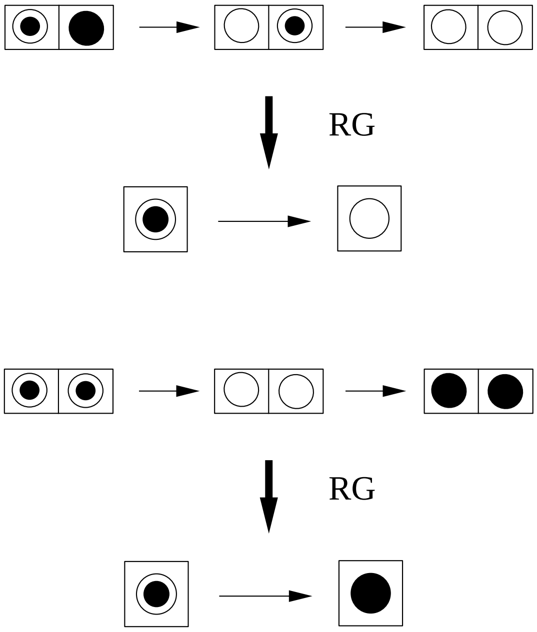


Figure 3: (a) Evolution of a burning cell into an empty cell in two time steps. (b) A possible proliferation in which a burning cell evolves in two time steps into a green cell. This process is of order $p^2$ and can therefore be neglected.

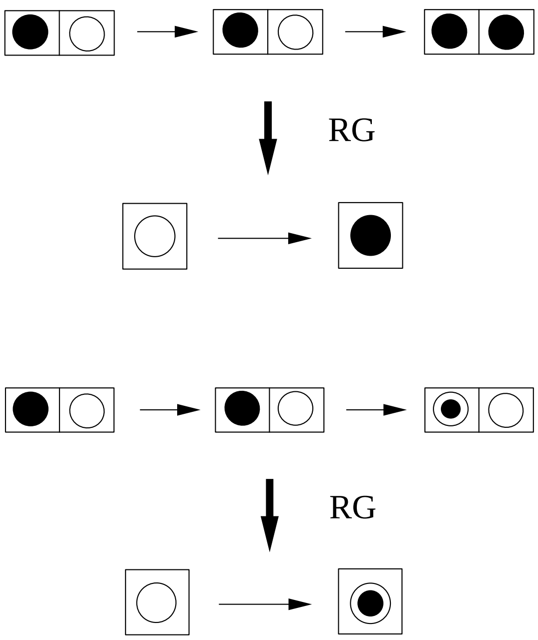


Figure 4: (a) A process contributing to the renormalization of $p$: an empty cell becomes green due to the growth of one site. (b) A possible proliferation of the growth dynamics in which an empty cell becomes burning. In the limit $f \ll p$ this process can be neglected.

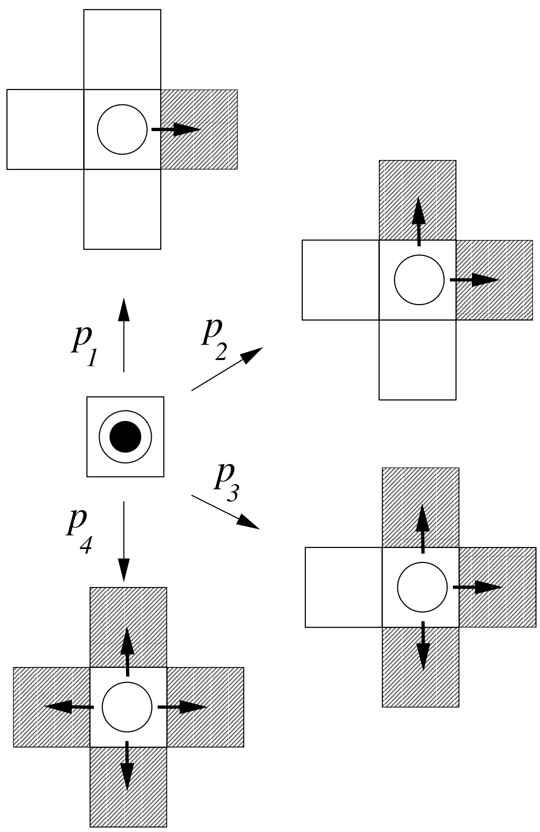


Figure 5: A possible proliferation in the sandpile dynamics.

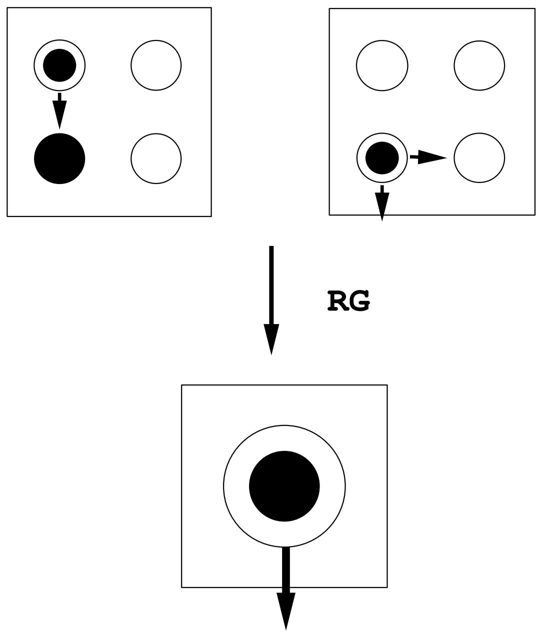


Figure 6: A path that contributes to the renormalization of $p_1$. See text for details.