Self-Structuring of Granular Media under Internal Avalanching

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We study the phenomenon of internal avalanching within the context of recently proposed "Tetris" lattice models for granular media. We define a recycling dynamics under which the system reaches a steady state which is *self-structured*, i.e., it shows a complex interplay between textured internal structures and critical avalanche behavior. Furthermore, we develop a general mean-field theory for this class of systems and discuss possible scenarios for the breakdown of universality.

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There has been a lot of interest in understanding the internal structure and geometry of granular packings [1]. The rich phenomenology observed in experiments studying compaction, segregation, and force distributions, among other things, has prompted a number of numerical and analytical studies. In another context the interest in granular media has also been triggered by the search for self-organized criticality (SOC) [2]. As a result, surface avalanches in granular piles have been extensively studied both experimentally [3] and in computer models [2,4] in order to identify clear regimes of SOC-like behavior [5]. In this Letter, we focus our interest on the interplay between the internal structure of packings and power-law avalanche distributions. We define a steady state dynamics under which the medium reaches a *self-structured critical* state. We also focus on the internal structure of this state and find, very interestingly, that it can be highly inhomogeneous with strong segregation and ordering effects.

The model we have investigated is a recently proposed simple lattice model for describing slow dynamical processes in granular media [6]. The basic ingredients of this model are the geometric constraints involved in packing particles of different shapes. This model is seen to reproduce experimentally observed phenomena such as slow relaxation in compaction [6], segregation [7], as well as aging [8].

Within the context of this model, we study the phenomenon of internal avalanches occurring under small perturbations. But as opposed to previous works [9], we focus on the stationary state that a system reaches under the continued process of removing a particle from the bottom layer and adding it back to the top of the system. Under this dynamics the system reaches a well defined "critical" steady state in which the avalanche distributions decay as power laws. Most interestingly, we find that in order to achieve this effect, the system restructures under this dynamics to a very inhomogeneous state with ordered regions (grains) separated by disordered low-density channels (grain boundaries) which act as preferential pathways for these avalanches. We perform our numerical experiments for particles of several different shapes and find that the steady state reached is always as described above, with an exponent for the power-law distribution which is the same for a large class of particle shapes. We furthermore develop a mean-field theory for systems undergoing this dynamics and explain, within this context, why we observe a universal power-law distribution. We elaborate on this point by considering a case when an important change in the rules of stability of particles changes the steady state reached and hence the universality class of the phenomenon.

We briefly review the definitions and some basic properties of the Tetris models [6] used in our simulations. Frustration arises in granular packings owing to excluded volume effects of particles of different shapes. This geometrical feature is captured in the Tetris model. In the following, we present results for the simplest version of the model where the particles are either rods with two kinds of orientations, more complicated shapes such as "T"- shaped particles with two kinds of orientations, or "crosses" with arms of randomly distributed lengths in the framework of the so-called random Tetris model (RTM) [10].

The Tetris model can be defined as a system of particles which occupy the sites of a square lattice tilted by 45° with periodic boundary conditions in the horizontal direction (cylindrical geometry) and a rigid wall at the bottom. Particles cannot overlap, and this condition produces very strong constraints (frustration) on their relative positions. This is illustrated for T-shaped particles in Fig. (1). In general each particle can be schematized as a cross with arms of different lengths which can be chosen in a regular [6] or in a random way [10]. The system is initialized by inserting the particles at the top of the system, one at a time, and letting them move down under gravity. The particles perform an oriented random walk on the lattice until they reach a stable position defined as a position from which they cannot fall any further because of other particles below them. The

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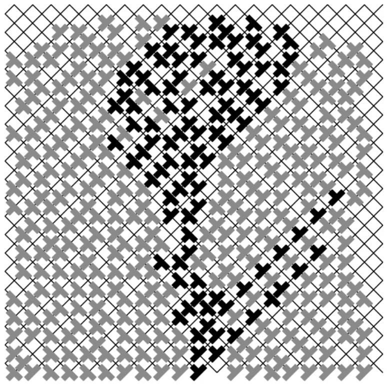

FIG. 1. A steady-state configuration of the Tetris model with T-shaped particles with two different orientations. The boundary conditions are periodic in the horizontal direction. The black particles are those which rearrange in the avalanche caused by removing the lowest particle.

particles retain their orientations as they move, i.e., they are not allowed to rotate. We now introduce the following dynamics under which the system evolves. A particle is removed from a random position at the base. This could destabilize its neighboring particles above one of which may then fall down if the geometry of the packing allows for the motion (i.e., if the orientation of the particle fits the local conformation). In this case, the disturbance propagates upwards destabilizing particles in the layer above and so on. We update the system sequentially, moving all the unstable particles down until the system is once more stable. The removed particle is then added back at a random position at the top of the system. This process is continued till the system reaches a steady state.

Similar procedures have been studied before for other models [11,12]. While long-ranged avalanche distributions have been found in [11], the update rule assumed in [12] does not lead to a critical state. We go beyond these previous works by studying here, in detail, the interplay between the avalanche distribution and the density profile of the medium. We explain the means by which the system reaches a critical state by developing a generic mean-field theory for avalanche distributions in dense or loose packings. We utilize the possibility afforded by this model, of easily changing particle shapes, to study this behavior for a wide variety of particle shapes. Most interestingly, we also find that this "critical" steady state is inhomogeneous and strongly ordered, different from those ordinarily studied in most SOC systems. These are thus some of the new features reported in the present study.

Figure 2a is a picture of a packing of in the steady state. As can be seen, it shows a complex textured structure. Namely, beginning from an initial state in which particles of different shapes are homogeneously mixed, the packing always "segregates" under the dynamics so as to form ordered high density grains separated by grain boundaries at lower densities. All avalanches preferentially propagate

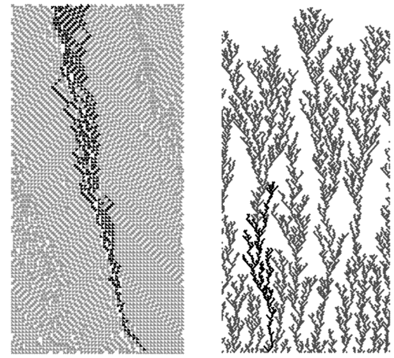

FIG. 2. Typical avalanches in the steady state for a system of T-shaped particles (left) and sticky particles (right).

inside these grain boundaries, i.e., no matter where the initial seed, the avalanches find their way into the boundary region (see Fig. 2, left).

The size of an avalanche is defined as the total number of particles destabilized by the process of removing a particle at the bottom. The size distribution of the avalanches decays like a power $P(s) \sim s^{-\tau}$. This was studied for the three different types of particles described above. Time averages were performed in the steady state over $\sim 10^6$ configurations in order to obtain good statistics. Figure 3 shows the avalanche distributions obtained for two different choices of the particles: the *T's* shown in Fig. 1 and particles with random shapes obtained in the framework of the RTM. In both cases one observes a scaling behavior for the avalanches with

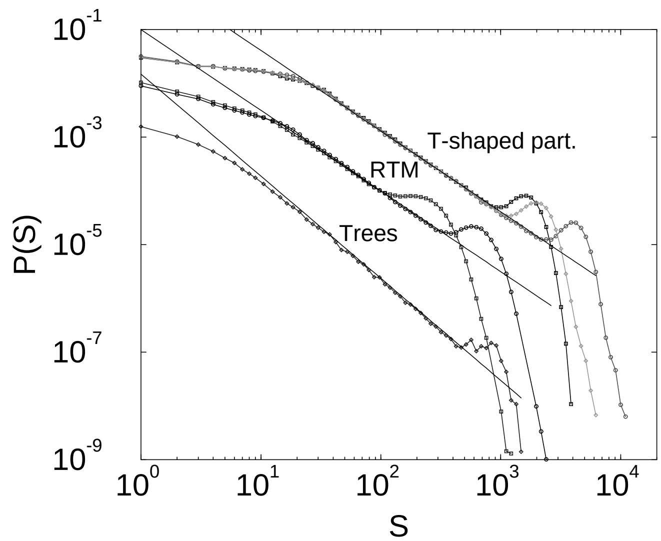

FIG. 3. $P(s)$ vs s in the steady state for T-shaped particles and the "crosses" (RTM): $\tau = 1.5 \pm 0.05$ in both cases. The system sizes shown are $Lx = 100, Ly = 500$ and $Lx = 200, Ly = 650, 1000$, respectively, for the "T's" and $Lx = 100, Ly = 150$, $Lx = 200, Ly = 300$ for the "crosses." The last curve shows the avalanche distribution for a system of sticky particles (see text). In this case one gets $\tau = 1.9 \pm 0.05$. The system size shown is $Lx = 200, Ly = 350$.

$\tau = 1.5 \pm 0.05$. In the case of the rods, the result is sensitive to the aspect ratio of the system, but for systems of about equal width and height the exponent of the avalanche distribution is again the same.

We now turn to a discussion of the avalanche statistics within the framework of a mean-field theory that we develop for this class of systems. It is apparent from our numerical studies reported so far that under this dynamics, the system reaches a steady state which is critical. The reason could be the following: Taking out a particle in the last layer creates a void in the packing. This can either move up (by exchanging place with a particle), die (if nothing above is destabilized), or free another neighboring void (and hence multiply) and propagate. The dynamics is thus essentially like a branching-annihilating process on the lattice where the probabilities for annihilating P_0, branching P_2, and propagation $P_1 = 1 - P_0 - P_2$ depend on the density of the packing. However, there is also a feedback effect. The avalanche distribution can in its turn affect the density of the system: large avalanches that reach the top tend to compactify the system and small avalanches make the system looser.

We can make the above arguments more precise in the following way. Let $\rho_h(t)$ be the cumulative density of the system up to height h at time t. Then the density of the system at time $t + 1$ will be

$$\rho_h(t + 1) - \rho_h(t) = -1/L^2 + a(t)h^{\gamma}/L^2\,, \qquad (1)$$

where L is the linear size (height) of the system. The first term of the right-hand side represents the effect of removing one particle. This is the sole contribution of avalanches which die before reaching the height h. On the other hand, those avalanches which reach at least a height h have the additional effect of increasing the density of the system by an amount equal to the number of voids which escape at h. This is equal to the width of the avalanche at h. If the avalanches are self-affine (as in [11], and also in the case studied here), i.e., an avalanche of height h has a width of h^{γ}, then the density increase is precisely given by the second term. The coefficient $a(t)$ is just a random variable which takes the value 1 whenever an avalanche reaches at least a height h and a value 0 otherwise. In the steady state, we can perform a time average on Eq. (1). We expect the left-hand side to vanish in this case. To evaluate the right-hand side, we note that the time average of $a(t)$ is simply $1 - \int_0^h P(h)\,dh$. We measure $P(h)$ numerically and observe the existence of a scaling region where $P(h) \sim h^{-\beta}$ with $\beta = 1.95 \pm 0.05$. We also independently measure γ (by measuring the characteristic size $s^* = h^{1+\gamma}$ of the avalanche size cut off at height h). We find $\gamma = 0.9 \pm 0.1$.

The above equation makes a prediction for the avalanche exponent. The steady state condition requiring that the average density of the system $\langle \rho \rangle = \mathrm{const}$ implies that $\beta - 1 = \gamma$. From the numerically measured value of β mentioned above, we see that we find $\gamma \sim 0.95$ consistent with the numerically measured value of γ. Making a change of variables from the avalanche height h to the avalanche size s gives us the relation $[1 - P_0(h)]h^{\gamma} = h^{\gamma}/h^{(1+\gamma)(1-\tau)} = 1$ where $P(s) \sim s^{-\tau}$ is the avalanche size distribution in the steady state. The above scaling relation for $\beta - 1 = \gamma$ then translates to (also obtained in [11] using a steady state argument)

$$\tau = 1 + \gamma/(1 + \gamma)\,. \qquad (2)$$

Using again our numerical estimate for γ we find $\tau \sim 1.47 \pm 0.05$ consistent with the data shown in Fig. 3.

A more complete and self-consistent description of the observed phenomenology can be obtained complementing Eq. (1) with an equation for the avalanche distribution $P(s)$ in terms of ρ, i.e., with an equation

$$P\big(s(t)\big) = F\big(\rho(t)\big)\,, \qquad (3)$$

where F indicates a generic function of $\rho(t)$. The two coupled equations (1) and (3) should then describe the evolution of the system to a steady state given by a critical density ρ_c with an avalanche distribution decaying as a power law. In general, it is difficult to write an exact equation for the avalanche distribution in terms of the density except for avalanches propagating on the Bethe lattice [13]. In this case, it is possible to show quite simply that the feedback effect of Eq. (3) on Eq. (1) results in the system reaching a critical density ρ_c given simply by the equation $P_1(\rho_c) + 2P_2(\rho_c) = 1$ where P_1 and P_2 are the probabilities for propagation and branching, respectively, introduced before. We have investigated analytically and numerically that the mean-field theory is insensitive to the exact functional form of the birth-death probabilities and avalanches always decay with an exponent $\tau = 1.5$ at the critical density. A more detailed analysis of the above equations considering different explicit forms of F is considered elsewhere [14].

The scaling relation (2) always holds for systems with open boundary conditions provided there is a compact bulk packing. This poses an upper limit to the exponent τ. For nonfractal bulk packings with a smooth free surface, γ cannot be larger than 1 and hence τ cannot be larger than 1.5. It is interesting to note that the avalanches decay with the same exponent as in mean-field theory. However, the reason for this exponent here is that the avalanches propagate in a conical region (implied by $\gamma \sim 1$) centered around the grain boundary (since, as mentioned before, avalanches propagate most of the time at the grain boundary). These facts imply, from the scaling relation (2), that $\tau = 1.5$.

Although our results have so far shown a universal behavior, we identify within the framework of this theory at least one clear instance of the breakdown of this universality. This has to do with having a very loose packing in the system. If this is the case, particles can fall large distances in the course of an avalanche and compactify the system far below. It would then not only be the width of the avalanche at height h which would contribute to the compactification but some fraction of the *whole* avalanche above h. Such an effect is clearly not taken into account in Eq. (1) which hence implicitly assumes that particles fall only short distances. We thus have to rewrite the above mean-field theory for a loose system for which the particles can fall large distances. We can quantify the above statements by rewriting Eq. (1) in the following manner:

$$\rho_h(t+1) - \rho_h(t) = -\frac{1}{L^2} + \frac{1}{L^2}\int_{s^*}^{\infty}(s - s^*)^{\alpha} s^{-\tau}\, ds\,, \tag{4}$$

where s^* is the typical size of an avalanche reaching a height h and α is a measure of how much of this avalanche contributes to heights less than h. Making a change of variables and taking the s^* dependence out of the integral, we find that the relevant scaling relation is now expressed in terms of s^* as $\tau = 1 + \alpha$. Since $\alpha \leq 1$ (if the total avalanche above h contributes, then $\alpha = 1$) we find that for systems with very loose packings the upper bound for the avalanche distribution is now $\tau \leq 2$ and not 1.5 as before.

We have checked this by changing the stability condition for particles to get a much looser packing. In all the cases considered above, the particles need to be stable in two directions in order not to fall. We modified this by looking at a system of *sticky* particles, in which one downward contact in either direction suffices for stability. Repeating the same recycling procedure used throughout the paper we find in this case a stationary state with a noncompact bulk packing (Fig. 2, right). For this system, we find an avalanche distribution in the steady state characterized by an exponent $\tau = 1.9 \pm 0.05$ (see Fig. 3) out of the range of validity of Eq. (2) and in the range of validity of the scaling relation predicted by Eq. (4).

There are several features that it is of interest to investigate further. An instability mechanism for producing structured steady states has yet to be developed [14]. Further, it would be interesting to see how these structures coexist with power-law avalanches and whether finite driving destroys this effect. In this context it can be seen from Fig. 3 that the big avalanches are enhanced well over the power law. It could be of interest to investigate whether this is just a finite-size effect or whether the structures play a role in this [14]. Within the context of this model we have also studied a system of spherical particles (i.e., crosses with roughly equal extensions in either direction). This is the case closest to the one studied in [11] in which the particles are all the same shape. We find that though a density plot is not sufficient to spot structures, an activity plot (marking how many avalanches pass through every site over a period of time) shows very distinctly that there are always long-lived loose regions where avalanches preferentially propagate with $\tau \sim 1.5$. It is hence tempting to conclude that this dynamics always results in long-lived inhomogeneities (with easy channels for particle flow) which affect the avalanche distribution. Finally, it is interesting to speculate what our results imply for possible experiments on the phenomenon of internal avalanches. One implication might be that a real system subjected to the continual process of removal and addition of grains will "fracture" (as in our model) developing easy regions for particle flow. It would be very interesting to see whether this is observable.

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