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What is the temperature of a granular medium?

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Abstract

In this paper we discuss whether thermodynamical concepts and in particular the notion of temperature could be relevant for the dynamics of granular systems. We briefly review how a temperature-like quantity can be defined and measured in granular media in very different regimes, namely the glassy-like, the liquid-like and the granular gas. The common denominator will be given by the fluctuation–dissipation theorem, whose validity is explored by means of both numerical and experimental techniques. It turns out that, although a definition of a temperature is possible in all cases, its interpretation is far from being obvious. We discuss the possible perspectives both from the theoretical and, more importantly, from the experimental point of view.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Granular matter [1–3] constitutes one of the most famous examples of non-equilibrium, athermal systems in which the usual temperature does not play any role. However, the fact that it involves many particles naturally brings a strong motivation to treat it with thermodynamic or thermodynamic-like methods. It is therefore of primary importance to determine whether such attempts are feasible, whether usual tools of thermodynamics can be generalized. In particular, can concepts like entropy or temperature have any meaning or use?

A thermodynamic approach is in general justified when one is able to identify a distribution that is left invariant by the dynamics (e.g. the microcanonical ensemble), and to assume that this distribution will be reached by the system, under suitable conditions of 'ergodicity'. Unfortunately, because energy is lost through collisions or internal friction, and gained by a non-thermal source (vibrations, tapping, shearing, . . .), the dynamical equations do not leave

the microcanonical or any other known ensemble invariant. This raises several questions, among which the most important is whether it is possible in principle to construct a coherent thermodynamics for these 'non-thermal' systems.

The concept of temperature has been in fact widely used in the context of the different states of granular media. The most straightforward definition comes from the case of dilute, strongly vibrated granular systems, which reach a non-equilibrium stationary state: by analogy with molecular gases, a 'granular temperature' T_g can be defined in terms of the average local kinetic energy per particle, and treated as one of the hydrodynamic fields. This approach can *a priori* be extended to denser, liquid-like, strongly vibrated systems.

In slowly evolving dense granular media, the situation is more complex since reaching a stationary state is experimentally or numerically very difficult. These systems actually exhibit ageing [4, 5] and memory [6, 7]. Analogies with other ageing systems have also led to the definition of dynamic temperatures as quantifying the violation of the equilibrium fluctuation–dissipation theorem (FDT) [8, 9].

In this paper, we will briefly review the different cases of dense, liquid-like and gas-like granular media, focusing on the notion of temperature as defined in the framework of the fluctuation–dissipation theorem.

The paper is organized along the following lines. In section 2 we review the fluctuation–dissipation theorem and its generalizations in order to provide the tools that will be used to test its validity in the different regimes. In section 3 we discuss the case of dense granular media. We first present a simple pedagogical 1D model for a granular medium where the notions of entropy and temperature emerge naturally and that could be used as a reference, although unrealistic, example. We then review the basics of Edwards' approach as well as some results obtained recently to test its validity. Next, in section 4 we describe experiments devoted to the test of the fluctuation–dissipation theorem in a vibro-fluidized granular medium, probed by means of a torsion pendulum. Section 5 is devoted to the discussion of the case of granular gases. Also in this case we shall present some procedures to test the validity of the fluctuation–dissipation theorem and to define a notion of temperature. Finally section 6 is devoted to the conclusions as well as to drawing some perspectives.

2. Fluctuation–dissipation theorem and generalizations

Let us consider an equilibrium system in contact with a thermostat at temperature T. For the purposes of this paper, the fluctuation–dissipation theorem [8] can be seen as relating the random diffusion and the mobility of a tracer particle in a gas or liquid: one possible version is the Einstein relation $\langle [r(t) - r(t')]^2 \rangle = 2\,\mathrm{d}T\,\frac{\delta\langle r(t)-r(t')\rangle}{\delta f}$, where r is the position of the particle and f is a constant perturbing field, and the brackets denote the average over realizations.

Since the FDT is a feature of equilibrium systems, nothing guarantees its validity in out of equilibrium systems. The situation is even worse in athermal systems, since it is not clear which quantity should play the role of the temperature.

Recent developments in theory of glasses, especially those related to their out of equilibrium dynamics, have shown that in slowly evolving, ageing systems the FDT is in fact modified in a very interesting fashion (for a recent review see [10]). In a class of mean-field models, which contains, although in a rather schematic way, the essentials of glassy phenomena [11, 12], and whose ageing dynamics was solved analytically [13], a feature that emerged was the existence of a temperature T_{dyn} for *all* the slow modes (corresponding to structural rearrangements) [14, 15]. This dynamical temperature T_{dyn} in fact exactly replaces the temperature of the heat bath in the Einstein relation; it is different from the external temperature, but it can be shown to have all other properties defining a true temperature [14].

In particular, the measure of $T_{\rm dyn}$ can be made using any version of the FDT relating the correlation and response of an observable [16]. Subsequently, the violation of FDT has become a widely studied tool in the context of glassy dynamics [17, 10].

The analogies between glassy thermal systems and non-thermal systems close to jamming [18] have stimulated investigations about the existence of dynamical or effective temperatures in non-thermal systems. In particular, the appearance of a dynamical temperature in models for dense, compacting granular media has been shown to arise from an Einstein-like relation [19–23]. We will focus on this aspect in section 3.

Other works on dense granular matter have focused on another version of the fluctuation–dissipation theorem which relates the energy fluctuations to the heat capacity [24–26]. Moreover, another athermal system has been investigated by Ono *et al* [27]. With reference to a model of sheared foam, various possible definitions of effective temperature have been shown to coincide, in particular in the context of an Einstein-like relation and of energy fluctuations.

On another side of the wide range of non-equilibrium systems, granular gases are very far from glassy systems. They are maintained in a dilute non-equilibrium steady state in which the dissipation due to inelastic collisions between particles is compensated by a strong energy injection. However, the existence of an Einstein relation between diffusion and mobility of a tracer particle has also been investigated recently in this context [28–31], as will be developed in section 5.

It should be remarked that up to now most of the work in this area has been theoretical and numerical and few experiments have been carried out in order to check the validity of the fluctuation–dissipation theorem in real granular media. One example in this direction is discussed in section 4, concerning a vibro-fluidized medium in a liquid-like regime [32].

3. Dense granular media

3.1. A pedagogical 1D model

In this section we consider a simple model which describes the evolution of a system of particles which hop on a lattice of $k = 0, \ldots, N$ stacked planes, as introduced in [33] and discussed in [34]. In particular, the system represents an ensemble of particles which can move up or down in a system of N layers in such a way that their total number is conserved. We ignore the correlations among particle rearrangements and problems related to the mechanical stability of the system. The master equation for the density ρ_k on a generic plane k, except for the $k = 0$ plane, is given by

$$\begin{aligned}\partial_t \rho_k = {} & (1-\rho_k) D(\rho_k)[\rho_{k-1} \cdot p_{\rm u} + \rho_{k+1} \cdot p_{\rm d}] + \\ & - \rho_k[(1-\rho_{k-1}) D(\rho_{k-1}) p_{\rm d} + (1-\rho_{k+1}) D(\rho_{k+1}) p_{\rm u}] \end{aligned} \tag{1}$$

where $p_{\rm d}$ and $p_{\rm u}$ (with $p_{\rm u} + p_{\rm d} = 1$) represent the probabilities for the particles to move downwards or upwards, respectively, among the different planes. $D(\rho_k)$ represents a sort of mobility for the particles given by the probability that the particle could find enough space to move. Apart from other effects it mainly takes into account the geometrical effects of frustration, i.e. the fact that the packing prevents the free movement of the particles. With $p_{\rm u}$ and $p_{\rm d}$ we can define the quantity $x = p_{\rm u}/p_{\rm d}$ which quantifies the importance of gravity in the system. We can associate with x a temperature for the system given by $T \sim 1/\log(1/x)$.

One interesting question to address is whether there exists a variational principle driving the relaxation phenomena in this system and in general in granular media. In other words one could ask whether some free-energy-like functional is minimized (Lyapunov functional) [35] by the dynamical evolution. In this model, and more generally for evolution equations of the

form

$$\partial_t \rho_k = g(\rho_k)[f(\rho_{k-1})p_\mathrm{u} + f(\rho_{k+1})p_\mathrm{d}] - f(\rho_k)[g(\rho_{k-1})p_\mathrm{d} + g(\rho_{k+1})p_\mathrm{u}] \quad (2)$$

where f and g are generic functions for which we only require $f \geqslant 0$, $g \geqslant 0$, $\mathrm{d}f/\mathrm{d}\rho \geqslant 0$, $\mathrm{d}g/\mathrm{d}\rho \leqslant 0$, it is possible to prove that such a functional, which decreases monotonically along the trajectories of the motion, indeed exists and is given by

$$F = \sum_{k=0}^{\infty} [\gamma(x)k\rho_k - S(\rho_k)], \quad (3)$$

with $S(\rho_k) = \int_{\rho_k} \log g(\rho)/f(\rho)\,\mathrm{d}\rho$ and $\gamma(x) = \log(1/x)$.

A deeper insight into the above mentioned phenomenology is obtained by considering the continuum limit for the model described by (2). More precisely, we consider a diffusive limit that consists in scaling the space variable as $1/\epsilon$, the time variable as $1/\epsilon^2$ and the drift term $p_\mathrm{d} - p_\mathrm{u}$ as ϵ. Therefore, $x = \epsilon k$, $\tau = \epsilon^2 k/2$, $p_\mathrm{d} - p_\mathrm{u} = \epsilon\beta/2$ and we consider the evolution of $u(x) \equiv \rho(k)$. We get the continuum limit by taking the Taylor expansion of the right-hand side of (2) around $x = k\epsilon$. For example, $\rho(k+1) \equiv u(x+\epsilon) = u(x) + \epsilon\partial_x u + \frac{1}{2}\epsilon^2 \partial_{xx} u + \mathrm{O}(\epsilon^3)$. We get, formally,

$$\partial_\tau u(x) = \beta\partial_x(fg) + (g\partial_{xx} f - f\partial_{xx} g) + \mathrm{O}(\epsilon), \quad (4)$$

which, in the limit $\epsilon \to 0$, gives

$$\partial_\tau u(x) = \beta\partial_x(fg) + (g\partial_{xx} f - f\partial_{xx} g). \quad (5)$$

This non-linear diffusion equation may be conveniently written in the following form:

$$\partial_\tau u = \partial_x \left(D(u)\partial_x \frac{\partial F}{\partial_\mathrm{u}} \right), \quad (6)$$

where $D = fg$, $\frac{\partial F}{\partial_\mathrm{u}}$ denotes the functional derivative of F with respect to u, $F = \int_0^\infty (\beta\ u\ x - S(u))\,\mathrm{d}x$, and $S' = \log(\frac{g}{f})$. Notice that the functional F decreases with the dynamics induced by (5). One has, in fact,

$$\partial_\tau F = \int \mathrm{d}x \frac{\partial F}{\partial u}\,\partial_\tau u = \int \mathrm{d}x \frac{\partial F}{\partial u}\,\partial_x \left(D(u)\partial_x \frac{\partial F}{\partial u} \right) \quad (7)$$

that, after an integration by parts, gives

$$-D(u)\left(\frac{\partial F}{\partial_\mathrm{u}} \right)^2 \leqslant 0. \quad (8)$$

Therefore there exists a 'free energy'-like functional, F, for (6) which has exactly the same form as the functional defined for the discrete model (see (3)). We can notice that, while the functional form of S and the value of β determine in a unique way the asymptotic state, they are not sufficient to determine the dynamical behaviour of the system. In particular, in order to know it one should know the functional form of $D(\rho)$.

What the analysis of this simple model suggests is the possibility of introducing, for non-thermal systems such as granular media, equilibrium concepts like free energy, entropy and temperature. More precisely, it is possible (in the case studied here) to predict the asymptotic state by means of the minimization of a suitable functional which can be constructed by entropic arguments. It is worth stressing how granular systems often exhibit memory and so the existence of a unique Lyapunov functional is not guaranteed in general. One could for example expect that several Lyapunov functionals are associated with different stationary states reached with different dynamical paths.

3.2. Edwards' approach

A very ambitious approach was put forward some years ago by Edwards and collaborators [36–39], by proposing for dense granular systems an equivalent of the microcanonical ensemble. The idea is to suggest that one could reproduce the observables attained dynamically by first measuring the density of the system, and then calculating the value of the observables in an ensemble consisting of all the *'blocked' configurations* at the measured density. The blocked configurations are defined as those in which every grain is unable to move.

This 'Edwards ensemble' leads naturally to the definition of an entropy $S_{\rm Edw}$, given by the logarithm of the number of blocked configurations of given volume, energy, etc, and its corresponding density $s_{\rm Edw} \equiv S_{\rm Edw}/N$. Associated with this entropy are state variables such as 'compactivity' $X_{\rm Edw}^{-1} = \frac{\partial}{\partial V} S_{\rm Edw}(V)$ and 'temperature' $T_{\rm Edw}^{-1} = \frac{\partial}{\partial E} S_{\rm Edw}(E)$.

That configurations with low mobility should be relevant in a jammed situation is rather evident; the strong hypothesis here is that the configurations reached dynamically are *the typical ones* of given energy and density. Had we restricted averages to blocked configurations having *all* macroscopic observables coinciding with the dynamical ones, the construction would exactly, and trivially, reproduce the dynamic results. The fact that conditioning averages to the observed energy and density suffices to give other dynamical observables (even if maybe only as an approximation) is highly non-trivial.

It turns out that the advances in glass theory mentioned in section 2 have in fact come to clarify and support such a hypothesis. Indeed, the dynamical temperature emerging from the Einstein-like relation between diffusion and mobility, despite its very different origin, matches exactly Edwards' ideas. One can indeed identify in mean-field models all the energy minima (the blocked configurations in a gradient descent dynamics), and calculate $1/T_{\rm Edw}$ as the derivative of the logarithm of their number with respect to the energy. An explicit computation shows that $T_{\rm Edw}$ coincides with $T_{\rm dyn}$ obtained from the out of equilibrium dynamics of the models ageing in contact with an almost zero-temperature bath [40–45]. Moreover, given the energy $E(t)$ at long times, the value of any other macroscopic observable is also given by the flat average over all blocked configurations of energy $E(t)$. Within the same approximation, one can also treat systems that, like granular matter, present a non-linear friction and different kinds of energy input, and the conclusions remain the same [46] despite the fact that there is no thermal bath temperature.

Edwards' scenario then happens to be correct within mean-field schemes and for very weak vibration or forcing. The problem that remains is to what extent it carries through to more realistic models. In the next subsections, we present the general methodology that has been used to explore this issue for some representative examples of models of granular compaction.

3.3. General strategy to check Edwards' assumption

One possibility of making an assumption *à la* Edwards would be to consider a fast quench, and then propose that the configuration reached has the macroscopic properties of the typical blocked configurations. This would imply that the system stops at a density for which the number of blocked configurations is maximal. However, it turns out that generically the vast majority of the blocked configurations are much less compact than the one reached dynamically, even after abrupt quenches.

One has therefore to give up trying to predict the dynamical energy or density by methods other than the dynamics itself. The strategy here is to quench the system to a situation of very weak tapping, shearing or thermal agitation. In this way, the system keeps compactifying, albeit at a very slow rate. Edwards' measure is then constructed as a flat measure over blocked configurations *conditioned to have the energy and/or density of the dynamical situation*.

In order to check Edwards' hypothesis, we will show how Edwards' measure can be constructed in some representative (non-mean-field) systems, together with the corresponding entropy and expectation values of observables. Moreover, we also construct what can be called Gibbs' measure by removing the constraint of sampling blocked configurations. Both measures are then compared with the observables obtained with an irreversible compaction dynamics.

3.3.1. Models. To illustrate the strategy to check Edwards' hypothesis we shall mainly refer to one specific finite-dimensional lattice model which has been shown to reproduce the complex phenomenology of the granular compaction. We focus on the so-called Kob–Andersen (KA) model [47], which considers a three-dimensional lattice gas with at most one particle per site and otherwise purely dynamical constraints: a particle can move to a neighbouring empty site only if it has strictly fewer than m neighbours in the initial and in the final position. This model was first devised in the context of mode-coupling theories to reproduce the cage effect existing in supercooled liquids, which produces at high density a very strong increase of the relaxation time[4]. Though very schematic, it has then been shown to reproduce rather well several aspects of glasses [49, 50] and of granular compaction [51]. The simplicity of its definition and the fact that it is non-mean-field makes it a very good candidate to test Edwards' ideas: in fact, the triviality of its Gibbs measure allows us to compare the numerical data obtained for the dynamics and for Edwards' measure with the analytic results for equilibrium.

Similar results have been obtained in the framework of another class of non-mean-field models, the so-called Tetris model [33, 5]. Here the constraints are not purely dynamical, but related to the steric properties of the grains which undergo a geometrical (and hence dynamical) frustration. We refer the reader to [20–22] for details.

3.3.2. Equilibrium measure. Let us first consider the case of the Kob–Andersen model: the dynamic character of the rule guarantees that the equilibrium distribution is trivially simple since all the configurations of a given density are equally probable: the Hamiltonian is just 0 since no static interaction exists. Density and chemical potential are related by $\rho = 1/(1 + \exp(-\beta\mu))$, and the exact equilibrium entropy density per particle reads

$$s_{\mathrm{equil}}(\rho) = -\rho \ln \rho - (1-\rho)\ln(1-\rho) \rightarrow \frac{\mathrm{d}s_{\mathrm{equil}}}{\mathrm{d}\rho} = -\beta\mu. \tag{9}$$

In this model, the temperature $1/\beta$, which appears only as a factor of the chemical potential, is irrelevant, so that we can set it to one. Besides, the equilibrium structure factor is easily seen to be a constant, $g_{\mathrm{equil}}(r) = \rho^2$: no correlations appear since the configurations are generated by putting particles at random on the lattice. It will therefore be easy, as already mentioned, to compare small deviations from $g_{\mathrm{equil}}(r)$, a notoriously difficult task to do in glassy systems.

3.3.3. Edwards' measure. Since Edwards' measure considers blocked configurations in which no particle is allowed to move, the crucial step to sample this measure is in fact to introduce an *auxiliary model* [19]: the auxiliary energy E_{aux} is defined as *the number of mobile particles*, where a particle is defined as mobile if it can be moved according to the dynamical rules of the original model. Edwards' measure is thus a flat sampling of the ground states ($E_{\mathrm{aux}} = 0$) of this auxiliary model, which is obtained by an annealing procedure, at fixed density, of the auxiliary temperature T_{aux} (and we write $\beta_{\mathrm{aux}} = 1/T_{\mathrm{aux}}$). Note that the

[4] For recent and interesting results about this model the reader is referred to [48].

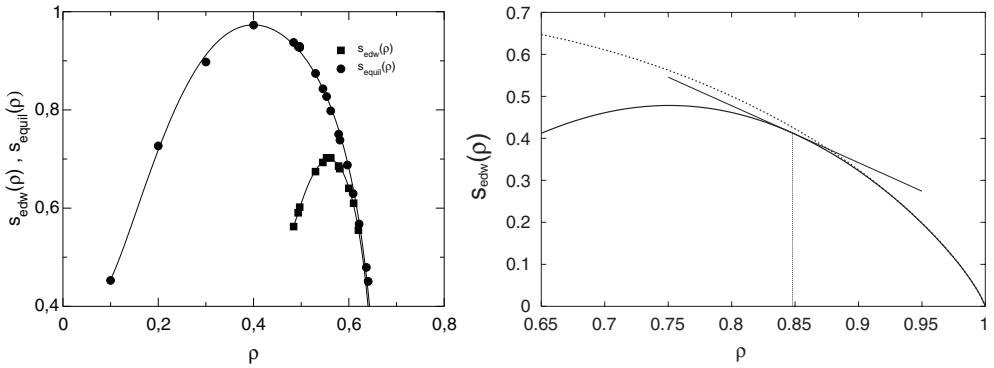

Figure 1. Edwards' entropy density, $s_{\rm Edw}(\rho)$, and equilibrium entropy density, $s_{\rm equil}(\rho)$, for the Tetris (left) (see [20] for details) and the Kob–Andersen (right) models.

Monte Carlo dynamics of the auxiliary model does not need to respect the constraints of the real model, so that efficient samplings with e.g. non-local moves can be obtained.

From the measure of the auxiliary energy during the annealing, at given density ρ, $E_{\rm aux}(\beta_{\rm aux}, \rho)$, one can compute the Edwards entropy density defined by

$$s_{\rm Edw}(\rho) \equiv s_{\rm aux}(\beta_{\rm aux} = \infty, \rho) = s_{\rm equil}(\rho) - \int_0^\infty e_{\rm aux}(\beta_{\rm aux}, \rho)\, {\rm d}\beta_{\rm aux} \tag{10}$$

where $e_{\rm aux}(\beta_{\rm aux}, \rho)$ is the auxiliary Edwards energy density and $s_{\rm equil}(\rho) = s_{\rm aux}(\beta_{\rm aux} = 0, \rho)$. Figure 1 reports the results for $s_{\rm Edw}(\rho)$ as obtained from (10) compared with $s_{\rm equil}(\rho)$, for the Tetris (left panel) and the Kob–Andersen models (right panel), respectively.

The slope of the tangent to $s_{\rm Edw}(\rho)$ for a generic ρ allows us to extract $T_{\rm Edw}(\rho)$. The natural definition for Edwards' temperature is

$$T_{\rm Edw}^{-1} = -\frac{1}{\mu}\frac{{\rm d}s_{\rm Edw}(\rho)}{{\rm d}\rho}; \tag{11}$$

which yields

$$T_{\rm Edw}(\rho) = \frac{{\rm d}s_{\rm equil}(\rho)}{{\rm d}\rho} \Big/ \frac{{\rm d}s_{\rm Edw}(\rho)}{{\rm d}\rho}. \tag{12}$$

Similarly, the Edwards measure structure function, $g_{\rm Edw}(r)$, is obtained as

$$g_{\rm Edw}(r) = \lim_{\beta_{\rm aux}\to\infty} g_{\rm aux}(r, \beta_{\rm aux}). \tag{13}$$

3.3.4. Irreversible compaction dynamics. The irreversible compaction dynamics is obtained by trying to increase the density of the system, starting from a low-density 'equilibrium' situation. For the KA model, this can be done e.g. by increasing slowly the chemical potential on a given layer of a three-dimensional box: when the chemical potential becomes large enough, the dynamical constraint no longer allows the system to reach the desired density and slow compaction follows.

During the compaction, we record the density $\rho(t)$ and the density of mobile particles $\rho_{\rm m}(t)$. It is particularly interesting to notice that in the out-of-equilibrium configurations visited during the irreversible dynamics the fraction of mobile particles is systematically smaller than the corresponding value in equilibrium. This suggests the possibility of distinguishing between equilibrium and out-of-equilibrium configurations by looking at the spatial organization of the

particles in both cases. We have thus measured, during the compaction dynamics, the particle–particle correlation function at fixed density.

The existence of an Einstein relation during the compaction dynamics is tested by the measure of the mobility of the particles

$$\chi(t_{\rm w}, t_{\rm w}+t) = \frac{1}{{\rm d}N}\sum_{a}\sum_{k=1}^{N}\frac{\delta\langle(r_k^a(t_{\rm w}+t) - r_k^a(t_{\rm w}))\rangle}{\delta f}, \tag{14}$$

obtained by the application of random forces to the particles between $t_{\rm w}$ and $t_{\rm w}+t$, and the mean square displacement

$$B(t_{\rm w}, t_{\rm w}+t) = \frac{1}{{\rm d}N}\sum_{a}\sum_{k=1}^{N}\left\langle(r_k^a(t_{\rm w}+t) - r_k^a(t_{\rm w}))^2\right\rangle, \tag{15}$$

(N is the number of particles; $a = 1, \ldots, d$ runs over the spatial dimensions: $d = 2$ for Tetris, $d = 3$ for KA; r_k^a is measured in units of the bond size d of the square lattice). Indeed, the quantities $\chi(t_{\rm w}, t_{\rm w}+t)$ and $B(t_{\rm w}, t_{\rm w}+t)$, at equilibrium, are linearly related (and actually depend only on t since time-translation invariance holds) by

$$2\chi(t) = \frac{X}{T_{\rm eq}}B(t), \tag{16}$$

where X is the so-called fluctuation–dissipation ratio (FDR) which is unitary in equilibrium. Any deviations from this linear law signals a violation of the fluctuation–dissipation theorem (FDT). In particular, as mentioned in section 2, in many ageing systems, and in particular in the KA model [50], violations from (16) reduce to the occurrence of two regimes: a quasi-equilibrium regime with $X = 1$ (and time-translation invariance) for 'short' time separations ($t \ll t_{\rm w}$), and the ageing regime with a constant $X \leqslant 1$ for large time separations. This second slope is typically referred to as a dynamical temperature $T_{\rm dyn} \geqslant T_{\rm eq}$ such that $X = X_{\rm dyn} = T_{\rm eq}/T_{\rm dyn}$ [14].

3.3.5. Comparing different measures. At this stage it is possible to compare equilibrium and Edwards' measures with the results of the out-of-equilibrium dynamics at large times.

In figure 2 we plot the deviations of the dynamical particle–particle correlation functions from the uncorrelated value ρ^2. In particular, we compare $\langle(g_{\rm dyn}(r) - \rho^2)\rangle$ obtained during the irreversible compaction with the corresponding functions obtained with the equilibrium and Edwards' measures. It is evident that the correlation function, as measured during the irreversible compaction dynamics, is significantly different from the one obtained with the equilibrium measure. On the other hand the correlation functions obtained with Edwards' measure are able to describe better what happens during the irreversible dynamics. In particular what is observed is that the correlation length seems to be smaller for configurations explored by the irreversible dynamics than in the equilibrium configurations. This aspect is captured by Edwards' measure which selects configurations with a reduced particle mobility. In practice, one can summarize the problem as follows: given a certain density, one can arrange the particles in different ways. The different configurations obtained in this way differ in the particle mobility and this feature is reflected by the change in the particle–particle correlation properties.

Another comparison can be performed with regard to the dynamical temperature $T_{\rm dyn}$ [50]. Figure 3 shows a plot of the mobility $\chi(t, t_{\rm w})$ versus the mean square displacement $B(t, t_{\rm w})$ testing in the compaction data the existence of a dynamical temperature $T_{\rm dyn}$ [50]. The agreement between $T_{\rm dyn}$ and the Edwards temperature $T_{\rm edw}$, obtained from the blocked

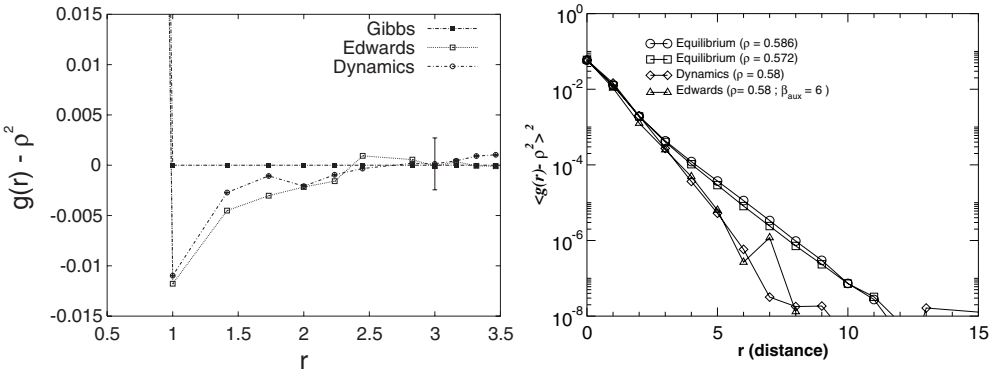

Figure 2. Left, **KA model**: dynamic structure function for the KA model, obtained in a very slow compression ('dynamics'), and with the equilibrium ('Gibbs') and Edwards measure at the density reached dynamically. Right, **Tetris model**: comparison between the correlation functions obtained with the equilibrium measure, the Edwards measure ($\beta_{\rm aux} = 6$) and the irreversible dynamics. In all cases the system is considered at a density of $\rho \simeq 0.58$.

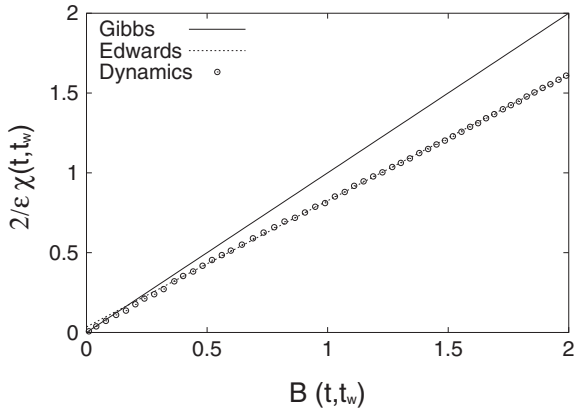

Figure 3. Einstein relation in the Kob–Andersen model: plot of the mobility $\chi(t, t_{\rm w})$ versus the mean-square displacement $B(t, t_{\rm w})$ (data shown as circles). The slope of the full straight line corresponds to the equilibrium temperature ($T = 1$), and the slope of the dashed one to Edwards' prescription obtained from figure 1 at $\rho(t_{\rm w}) = 0.848$.

configurations as in figure 1, for the density at which the dynamical measurement were made, is clearly excellent. Further evidence in this direction has been obtained for the Tetris model [21].

To summarize, during the compaction, the system falls out of equilibrium at high density, and is therefore no longer described by the equilibrium measure. It turns out that Edwards' measure, constructed by a flat sampling of the blocked configurations at the dynamically reached density, reproduces the physical quantities measured at large times, and in particular predicts the correct value for the dynamical temperature.

3.4. Partial conclusions

It turns out that Edwards' measure, constructed by a flat sampling of the blocked configurations, is able to reproduce the physical quantities measured at large times. In particular, the connection of Edwards' temperature with the dynamical FDT temperature seems to be generally valid (though there are important counterexamples [20, 52]).

Apart from the first evidence reviewed here, which has lent credibility to Edwards' construction, various works have allowed a better comprehension of the validity of Edwards' approach and of its limitations. A comprehensive review of these approaches is beyond the scope of the present paper. An incomplete list would include results on 1D models of particles or spins [24, 25, 53], parking lot models [54], 3D molecular dynamics simulations [23], diluted spin-glasses and hard spheres on lattice [26], spin-glasses with tapping [55].

It is important to note that, in a case for which the explicit analytical computation of Edwards' measure and of the dynamical quantities is feasible, it has been shown that Edwards' construction of a flat measure on the blocked configurations is not exactly valid [52]. In [20], the study of the low-temperature dynamics of the random field Ising model has also shown that the dynamically reached configurations have typically zero magnetization, in contrast to the configurations dominating Edwards' measure. While such studies show that Edwards' construction should probably be considered only as a first approximation, generalizations have been shown to yield better and better results for the prediction of observables obtained dynamically. These generalizations imply to use a flat average on blocked configurations, restricted by constraining more than just one variable to its value obtained dynamically [56, 22]. The question arising then concerns the number of observables to constrain in order to obtain reliable predictions for the others.

While such theoretical aspects remain interestingly open, a crucial question concerns the experimental validation of Edwards' ideas. In particular, the study of diffusion and mobility of different tracer particles within driven granular media would allow us to confirm or disprove the theoretically predicted violation of FDT and the existence of dynamical temperatures[5]. Such experiments are actually in progress and a first set of results concerning the diffusion has been published in [58, 59].

4. Liquid-like granular media

4.1. Context

In this section we analyse whether a notion of temperature can be defined for a granular medium in a liquid-like regime, i.e. a regime where the medium is brought by vibration to a quasi-fluidized state. We shall mainly refer to some recent experimental results where a vibrating granular medium was sensed by means of a torsion pendulum.

In the classical Brownian motion experiment, a 'tagged' particle immersed in a liquid can be used as a thermometer to determine the temperature of the liquid itself. For this, one has to record the motion of the tagged particle, and data analysis, for example according to the Langevin formalism, gives the temperature. One may wonder whether a similar experiment, performed in a granular medium under suitable external vibrations (so that it looks very much like a liquid), could be used to determine a 'granular-liquid temperature'.

4.2. Theoretical background

Let us briefly review for clarity the behaviour of a torsion oscillator of moment of inertia I and elastic constant G immersed in an equilibrium liquid. Following the Langevin hypothesis [60], we suppose that the effect of this perturbing environment is split into two parts: a viscous friction force, proportional to the oscillator angular velocity, and a random, rapidly fluctuating

[5] For alternative experimental tests for the validity of Edwards' approach, inspired by the study of spin systems, see [57].

force $\xi(t)$, which is an uncorrelated Gaussian white noise of zero mean and variance q. The oscillator angular position θ satisfies the Langevin equation:

$$I\ddot{\theta}(t) + \alpha\dot{\theta}(t) + G\theta(t) = \xi(t) + C_{\text{ext}}(t) \tag{17}$$

where α is a friction coefficient and C_{ext} denotes an external torque to which the system may also be subjected. When no external torque is applied ($C_{\text{ext}} = 0$), a useful quantity that can be extracted from this equation is the power spectral density S, defined as twice the Fourier transform of the auto-correlation function $\langle\theta(t)\theta(t')\rangle$ (where $\langle\cdots\rangle$ denotes the statistical average over the noise). Using the Wiener–Khintchine theorem for stationary processes, we get

$$S(\omega) = \frac{2q}{I^2(\omega^2 - \omega_0^2)^2 + \alpha^2\omega^2} \tag{18}$$

where $\omega_0 = \sqrt{G/I}$ is the natural angular frequency of the oscillator.

On the other hand, one can also focus on how the oscillator responds to an external torque $C_{\text{ext}}(t)$. The quantity containing this information is the susceptibility $\chi(t)$ (or linear response function), defined as

$$\theta(t) = \int \mathrm{d}t'\, \chi(t - t')C_{\text{ext}}(t') \tag{19}$$

which implies that the external torque $C_{\text{ext}}(t)$ should be small enough for this linear approximation to be valid. From this definition and from the Langevin equation (17), we see that in the Fourier representation

$$\chi(\omega) = \frac{\theta(\omega)}{C_{\text{ext}}(\omega)} = \frac{1}{I\left(\omega_0^2 - \omega^2\right) + \mathrm{i}\alpha\omega}. \tag{20}$$

The real and imaginary parts of this complex function can be defined as $\chi(\omega) = \chi'(\omega) - \mathrm{i}\chi''(\omega)$, where in particular

$$\chi''(\omega) = \frac{\alpha\omega}{I^2(\omega^2 - \omega_0^2)^2 + \alpha^2\omega^2}. \tag{21}$$

Comparing the power spectral density (18) and the imaginary part of the susceptibility (21), we now notice that these two very different concepts have similar expressions and are related by the simple relation

$$\frac{S(\omega)\omega}{\chi''(\omega)} = \frac{2q}{\alpha}. \tag{22}$$

In a thermal system at equilibrium, using the equipartition of energy principle, the parameter q can be related to the bath temperature as $q = 2\alpha k_{\text{B}}T$, thus giving the celebrated fluctuation–dissipation theorem which states that

$$S(\omega)\omega/\chi''(\omega) = 4k_{\text{B}}T. \tag{23}$$

Since a vibrated granular medium is not at equilibrium, there is no reason, in principle, to expect that a relation like (23) should hold for such a system. We can expect that a granular medium can be found in quasi-stationary states but no ergodic principle can be invoked whatsoever. Nevertheless, if a simple relation like (23) were valid for a granular medium, at least in some regime, this would be a strong hint for the comprehension of the thermodynamic properties of such systems.

The experiment we present was aimed precisely at the check of relation (23). The idea is to measure the noise power spectrum and the susceptibility while the granular medium is externally driven in the liquid-like state, and test whether there exists a fluctuation–dissipation-like relation. If so, this will give us a measure of a temperature-like parameter.

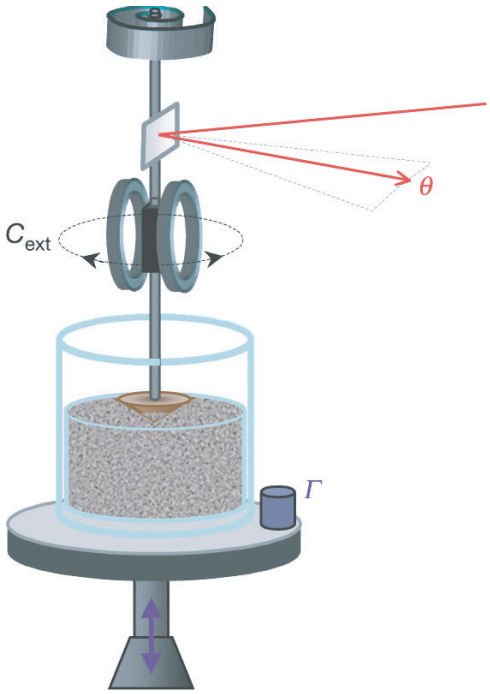

Figure 4. Sketch of the torsion oscillator immersed in the 'granular bath'. The granular medium, composed of glass beads of diameter 1.1 ± 0.05 mm, is placed in a cylindrical container of height 60 mm and diameter 94 mm. An accelerometer measures the intensity of the external perturbations, Γ.

4.3. Experimental set-up

We use the following experimental set-up [32]: a thin torsion oscillator is immersed at some depth in a granular medium made of millimetre-size glass beads, as shown in figure 4 (note the analogy with the situation described in [61], for a system at thermal equilibrium). The beads are placed into a cylindrical container which is continuously vibrated vertically, with a high-frequency filtered white noise (cut off above 900 Hz and below 300 Hz in the experiments described). We use this vibration mode to ensure a homogeneous agitation and avoid undesired effects such as pattern formation and convection rolls. Note that this type of white noise vibrations is not used in order to provide a random torque with white noise spectrum to the oscillator: actually, its motion is observed in a much lower frequency range (10–50 Hz) than the vibrations applied.

The vibration intensity is determined by an accelerometer fixed on the container, which measures the parameter Γ, defined as $\Gamma^2 = \int A(f)\,\mathrm{d}f$, where $A(f)$ is the acceleration spectrum, normalized to the acceleration of gravity, and the integration is taken in the frequency range of about 1 Hz to 10 kHz. For sinusoidal vibrations, $\Gamma = 1$ is the 'fluidization' threshold, above which a single grain starts to fly. Here, we typically use vibration intensities between $\Gamma = 1$ and 15.

The oscillator angular position θ is detected optically (see figure 4). For susceptibility measurements, two external coils and a permanent magnet fixed on the oscillator axis allow us to apply a sinusoidal torque $C_{\mathrm{ext}}(t) = C_{\mathrm{e}} \sin(\omega t)$.

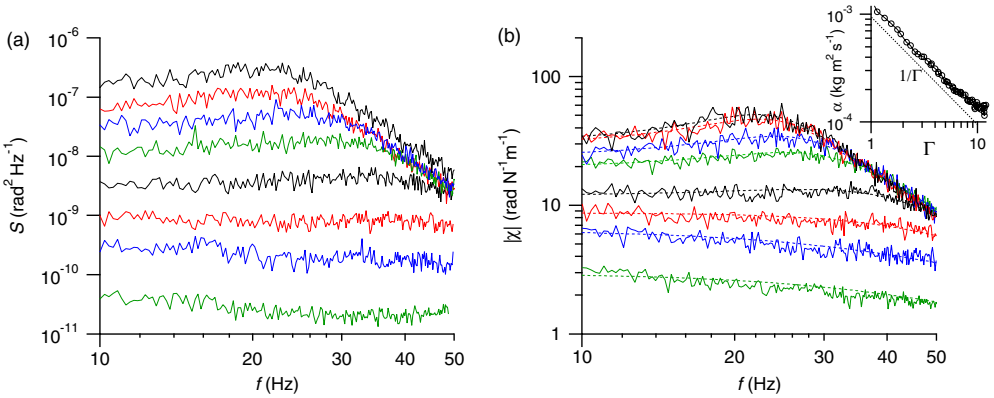

Figure 5. Power spectral density (a) and modulus of the complex susceptibility (b) versus the frequency, $f = \omega/2\pi$, for different vibration intensities Γ: from top to bottom, 11.6, 10.0, 7.3, 5.4, 3.7, 2.2, 1.5, 1.0. Results obtained with a conical probe of apex angle 120°, covered with a single layer of glass beads. The immersion depth is about 11 mm. In (b), each curve is fitted (dashed line) with its corresponding equilibrium expression (see text for details). The moment of inertia of the oscillator is $I = 1.5 \times 10^{-6}$ kg m^2, and the applied torque amplitude is $C_e = 3.2 \times 10^{-5}$ N m. Inset: friction coefficient α, obtained from the fit to the curves $|\chi(\omega)|$, versus Γ. The dotted line is a power law $\alpha \propto 1/\Gamma$.

4.4. Results and discussion

The analysis of angular deflection time-series $\theta(t)$ in the free mode ($C_{\rm ext} = 0$) provides the noise power spectral density $S(\omega)$, shown in figure 5(a) for different values of Γ. Then, with the oscillator in forced mode (with an externally applied torque $C_{\rm ext}(t)$), we measure the complex susceptibility $\chi(\omega)$, whose modulus $|\chi(\omega)|$ is shown in figure 5(b). The amplitude of the external torque is small enough to be in the regime of linear response.

Fitting these curves $|\chi(\omega)|$ with the standard expression for the damped oscillator $|\chi(\omega)| = [I^2(\omega_0^2 - \omega^2)^2 + \alpha^2\omega^2]^{-1/2}$ shows a good agreement, thus supporting the idea that the Langevin equation of motion (17) is a pertinent description of the oscillator linear response. This allows us to extract a granular friction coefficient α, or a granular viscosity $\mu \propto \alpha$, found to be inversely proportional to the vibration intensity: $\alpha \propto 1/\Gamma$ (see inset of figure 5(b)).

We can also calculate the fluctuation–dissipation ratios $S(\omega)\omega/(4\chi''(\omega))$, which are shown in figure 6(a). These ratios, even though not constant, are surprisingly 'flat', especially compared to what has been measured in other non-equilibrium thermal systems, such as in glycerol [62] and laponite [63].

This reveals that the high-frequency driven agitation of the granular medium acts on the oscillator as a source of random torque with white spectrum, at least in the 10–50 Hz range under consideration. Energy is thus injected at high frequency, and spreads into a low frequency white spectrum.

Since these ratios do not exhibit a strong frequency dependence, this provides support for the existence of a fluctuation–dissipation relation in off-equilibrium driven granular steady states. This relation can thus be used to define an effective temperature, $T_{\rm eff}$. Figure 6(b) shows the averaged fluctuation–dissipation ratio levels, that is, $k_B T_{\rm eff}$, versus Γ (black curve). Fitting to a power law yields $k_B T_{\rm eff} \propto \Gamma^p$ with p close to 2. This dependence suggests that, due to the complex dissipation processes between the grains, a fixed fraction of the energy input (vibrations) is effectively available as granular kinetic energy and is 'sensed' by the oscillator.

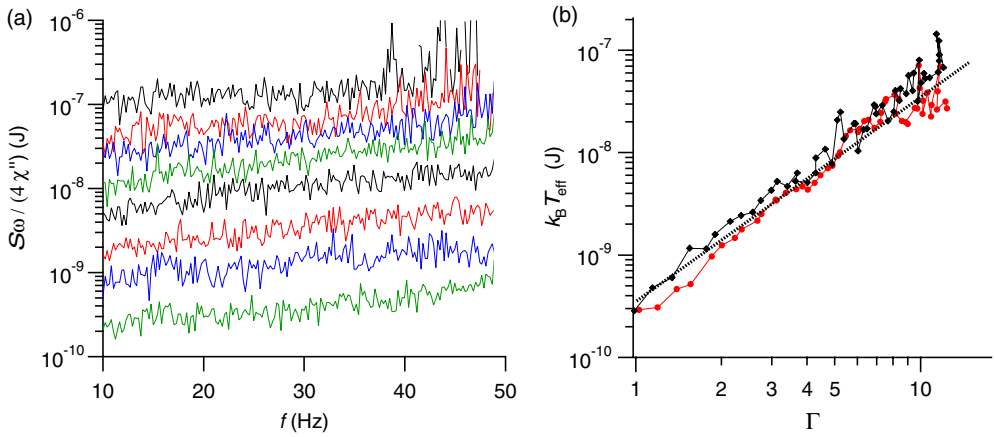

Figure 6. (a) Fluctuation–dissipation ratios $S(\omega)\omega/(4\chi''(\omega))$ versus the frequency $f = \omega/2\pi$ for different vibration intensities (as in figure 5). (b) Effective temperature versus vibration intensity Γ, as obtained from fluctuation–dissipation levels in (a) averaged between 10 and 50 Hz (black diamonds), and from experiment using a conical probe with triple moment of inertia (red circles). A power law fitted to the data gives $T_{\rm eff} \propto \Gamma^p$, with $p = 2.1$. The dotted line has equation $k_{\rm B}T_{\rm eff} = 3.5 \times 10^{-10}\Gamma^2$.

In fact, we notice that the order of magnitude of the thermal energy $k_{\rm B}T_{\rm eff}$, as measured here, is consistent with realistic values of the mean kinetic energy per particle, as measured by grain-tracking methods [64, 65]. Thus, the effective temperature $T_{\rm eff}$ measured seems to be related to the granular temperature, as usually defined in granular gases. This is particularly interesting in view of the recent numerical experiments relating the granular temperature with a dynamical temperature obtained through FDT-like measures in granular gases (see section 5 and [29, 30]).

These results indicate that the use of macroscopic quantities, such as temperature-like or viscosity-like parameters, to describe the behaviour of externally driven granular media is in first approximation possible. We can notice that, as shown in figure 6(b), the temperature parameter obtained does not seem to depend on the moment of inertia of the probe used. On the other hand, we also have indications (not discussed in detail here) that several complications, related to the discrete and inhomogeneous nature of granular media and to the role of gravity, have to be taken into account. These mechanisms are triggered as soon as the probe–granular medium interactions take place on length scales that are comparable to the grain dimensions, for instance when using non-smooth probe sections exhibiting features with length scales of the same order as the grain diameter, or close to the surface . . .. The systematic study of these geometrical and grain-level effects will give us the possibility to establish empirical laws that may be used to predict granular behaviour in practical situations, and possibly in simulations.

Let us note that the temperature measured here is not *a priori* the Edwards temperature. However, by decreasing the external vibrations we observe evidence of a glassy behaviour [66]. In particular, the study of the power spectrum (hence of the diffusivity) reveals that for very weak external vibrations the granular medium exhibits a critical slowdown from the fluid state to a glassy state, where—as for supercooled liquids—the diffusivity approaches zero. In thermal systems undergoing a glass transition, the definition of a temperature is an open issue, and there are suggestions proposing that a 'configurational temperature', sharing similarities with the Edwards temperature, can be introduced. Thus, in the limit of very small external vibrations, it is possible that we are confronted with a similar problem in the granular media.

5. Granular gases

In this section the particular case of highly fluidized granular materials is taken into account. When a box full of grains is strongly shaken and the volume available is large enough, the assembly of grains behaves in a way similar to standard molecular gases. In the literature a large series of experiments has led to the definition of 'granular gases', i.e. granular material in a gaseous state [67, 68]. Because of the analogy with usual gases, the term 'granular temperature' has naturally been defined as the average local kinetic energy per particle: this definition indeed coincides with the thermodynamic equilibrium temperature in the limit of elastic collisions between particles.

In experiments, it is possible to study kinetic as well as hydrodynamic observables and compare them with the results of statistical mechanics and hydrodynamics of molecular gases. The observed differences may be explained as a consequence of loss of energy in collisions between grains. Energy dissipation during a collision is due to irreversible transfer of energy from macroscopic energy to internal degrees of freedom and eventually to heat. Therefore it can be said that *inelasticity* is the main ingredient in the description of a granular material in the dilute regime, while frustration and excluded volume effects are negligible. We will show that, while inelastic collisions produce large deviations with respect to usual thermodynamics and hydrodynamics, linear response theory and fluctuation–dissipation relations are still valid [29–31], provided that the 'equilibrium temperature' is replaced by the granular temperature of each component of the gas.

5.1. Fluctuation–dissipation relations for diluted granular systems

We have studied fluctuation–dissipation relations using two sets of independent measurements, i.e. two choices of the pair response correlation.

Recipe I. The first one consists in the classical measure of mobility and diffusivity. The mean-square displacement

$$B(t,t') = \frac{1}{N}\sum_{j=1}^{N}\langle|\mathbf{r}_j(t) - \mathbf{r}_j(t')|^2\rangle, \tag{24}$$

behaves asymptotically as $\sim 4D(t-t')$. The mobility of a tracer particle can be measured by applying a constant and small drag force $\boldsymbol{\xi} = \xi\mathbf{e}_x$ to a given particle, labelled 0, for times $t > t'$ (linearity of the response has been checked by changing the amplitude of the perturbation). The perturbed particle will reach at large times a constant velocity μ, related to the response by

$$\chi(t,t') = \frac{1}{\xi}\langle(\mathbf{r}_0(t) - \mathbf{r}_0(t'))\cdot\mathbf{e}_x\rangle \approx \mu t \qquad \text{at large } t. \tag{25}$$

At *equilibrium* Green–Kubo relations (the Einstein relation in this case) predict $\mu = \beta D$, $T_b = 1/\beta$ being the equilibrium temperature, so that $\chi(t,t') = \frac{\beta}{4}B$.

Recipe II. Another totally independent way of checking FD relations is the following: once a steady state has been reached, the system is perturbed impulsively at a given time t_0 by a non-conservative force applied (non-uniformly) on every particle. The response is then monitored at later times. The force acting on particle i is

$$\mathbf{F}(\mathbf{r}_i,t) = \gamma_i\boldsymbol{\xi}(\mathbf{r}_i,t) \tag{26}$$

with the properties $\nabla \times \boldsymbol{\xi} \neq 0$, $\nabla \cdot \boldsymbol{\xi} = 0$, where γ_i is a particle dependent variable with randomly assigned ± 1 values. A simple case is realized by a transverse perturbation $\boldsymbol{\xi}(\mathbf{r}, t) = (0, \xi \cos(k_x x)\delta(t))$, where k_x is compatible with the periodic boundary conditions, i.e. $k_x = 2\pi n_k / L_x$ with n_k integer and L_x the linear horizontal box size. The staggered response function R (i.e. the current induced at t by the perturbation at t_0), and the conjugated correlation C,

$$R(t, t_0) = \frac{1}{\xi} \left\langle \sum_i \gamma_i \dot{y}_i(t) \cos(k_x x_i(t)) \right\rangle,$$

$$C(t, t_0) = \left\langle \sum_i \dot{y}_i(t) \dot{y}_i(t_0) \cos\{k_x [x_i(t) - x_i(t_0)]\} \right\rangle$$

are related, *at equilibrium*, by the FD relation $R(t, t_0) = \frac{\beta}{2} C(t, t_0)$, $T_b = 1/\beta$ being the equilibrium temperature.

We have applied both recipes to a model of pure granular gas as well as to a model of granular binary mixture. We have also applied recipe I to a much simplified model of pure granular gas (inelastic Maxwell model) where analytical calculations of diffusion and mobility can be straightforwardly obtained.

5.2. Description of the models

The simplest model of granular material in two dimensions is the hard disc gas with inelastic collisions. To counterbalance the loss of energy we enforce the stochastic forcing model, i.e. particles receive random acceleration as if they were in contact with a 'heat bath'. Moreover, we may (or may not) provide a viscous drag to each particle. Viscosity has the role, in this model, of a regularizing force and allows for a better definition of elastic limit. It can be thought as the result of friction with external walls or with a fluid the gas is immersed in. However we will show that viscosity is not essential in this study and identical results can be obtained without it.

We consider a volume V in dimension $d = 2$ containing $N = N_1 + N_2$ inelastic hard discs, N_1 and N_2 being the number of particles in component 1 and 2 of the mixture, respectively. The discs have diameters σ (identical for the two species) and masses m_{s_i} (where $1 \leqslant i \leqslant N$ and s_i is the species index, 1 or 2, of particle i). In a collision between spheres i and j, characterized by the inelasticity parameter called the coefficient of normal restitution $\alpha_{s_i s_j}$, the pre-collisional velocity of particle i, $\mathbf{v}_i$, is transformed into the post-collisional velocity $\mathbf{v}_i'$ such that

$$\mathbf{v}_i' = \mathbf{v}_i - \frac{m_{s_j}}{m_{s_i} + m_{s_j}} (1 + \alpha_{s_i s_j})(\hat{\sigma} \cdot \mathbf{v}_{ij}) \hat{\sigma} \tag{27}$$

where $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$ and $\hat{\sigma}$ is the centre to centre unit vector from particle i to j ($\alpha_{s_i s_j} = \alpha_{s_j s_i}$ so that the total linear momentum $m_i \mathbf{v}_i + m_j \mathbf{v}_j$ is conserved).

In between collisions, the particles are subjected to a random force in the form of an uncorrelated white noise (e.g. Gaussian) with the possible addition of a viscous term. The equation of motion for a particle is then

$$m_i \frac{\mathrm{d}\mathbf{v}_i}{\mathrm{d}t} = \mathbf{F}_i + \mathbf{R}_i - \zeta_{s_i} \mathbf{v}_i \tag{28}$$

where $\mathbf{F}_i$ is the force due to inelastic collisions, ζ_{s_i} is the viscosity coefficient and $\langle R_{i\alpha}(t) R_{j\beta}(t') \rangle = \xi_{s_i}^2 \delta_{ij} \delta_{\alpha\beta} \delta(t - t')$, where Greek indices refer to Cartesian coordinates. The granular temperature of species s is given by its mean kinetic energy $T_s = m_s \langle v^2 \rangle_s / d$ where $\langle \cdots \rangle_s$ is an average restricted only to particles of species s.

Pure system. When $m_1 = m_2 = m$, $\zeta_1 = \zeta_2 = \zeta$, $\xi_1 = \xi_2 = \xi$ and $\alpha_{11} = \alpha_{12} = \alpha_{22} = \alpha$, the gas is monodisperse. This model has been extensively studied [69–76]. When $\zeta \neq 0$ a 'bath temperature' can be defined as $T_\mathrm{b} = \xi^2/2\zeta$. This corresponds to the temperature of a gas obeying equation (27) with elastic collisions or without collisions. The same temperature can be observed if the viscosity is very high, i.e. when $\zeta \gg 1/\tau_\mathrm{c}$ where τ_c is the mean free time between collisions. Here we recall that when $\alpha < 1$ the gas still attains a stationary regime, but its granular temperature is smaller than T_b and therefore the system is out of equilibrium. Moreover, the statistical properties of the gas are different from those of an elastic gas in contact with a thermal bath: mainly the velocity distribution is non-Gaussian with enhanced high-energy tails. The system is usually studied by means of molecular dynamics (with hard core interactions) or by means of numerical solutions of the associated Boltzmann equation. As we are interested in the dilute case, where molecular chaos is at work, we follow this second recipe, implementing the so-called direct simulation Monte Carlo [77] (DSMC) which is a numerical scheme that solves the Boltzmann equation for homogeneous or (spatially) non-homogeneous systems.

Binary mixture. In the general binary mixture case, simulations as well as experiments and analytical calculations have shown that energy equipartition is broken, i.e. $T_1 \neq T_2$. At the level of the Boltzmann kinetic equation, the temperature ratio of a binary granular mixture subject to stochastic driving of the form given above has been obtained in [78] for the case $\zeta_{s_i} = 0$ and in [79] for $\zeta_{s_i} \neq 0$. In the case $\zeta_{s_i} \neq 0$ a bath temperature can still be defined as $T_\mathrm{b} = \xi_{s_i}^2/2\zeta_{s_i}$. Note that in general ξ_{s_i} and ζ_{s_i} depend on m_i and the correct elastic limit (i.e. equipartition) is recovered if and only if T_b does not depend upon m_i. In [79] it has been shown that a model with $\xi_{s_i} \propto \sqrt{m_{s_i}}$ and $\zeta_{s_i} \propto m_{s_i}$ reproduces fairly well experimental results for the temperature ratio T_1/T_2 measured in a gas of grains in a vertically vibrated box. It is also known that equipartition is not recovered even in the so-called tracer limit [80], i.e. in the case $N_2 = 1$ and $N_1 \ll 1$. For binary mixtures we have implemented both molecular dynamics and DSMC, in order to study possible differences.

Inelastic Maxwell model. In analogy with elastic gases, a simplified model can be introduced. For instance, the inelastic analogue of a Maxwell gas [82] has recently been proposed [81]. An elastic Maxwell gas is made by particles interacting through a special repulsive long range potential. The interest of such an interaction is that the corresponding collision frequency turns out to be strongly simplified: while for hard discs the collision frequency is proportional to the relative impact velocity g, for particles of a Maxwell gas it becomes independent of g. So the resulting Boltzmann equation takes a much simpler convolutive expression. The stochastic model of inelastic Maxwell gases is directly defined, introducing a normal restitution coefficient smaller than one in the kinetic equation of an elastic Maxwell gas. As in the elastic case, the interest for such an operation is the achievement of interesting exact results.

In the simplest case, i.e. in one dimension, the Boltzmann equation of an inelastic Maxwell gas reads $\partial_t P(v,t) = \beta \int \mathrm{d}u\, P(u,t) P(\beta v + (1-\beta)u, t) - P(v,t)$, where $\beta = 2/(1+\alpha)$ and α is the restitution coefficient. At odds with the elastic case, its only stationary solution is the degenerate delta function, representing a system of particles at rest. Starting with a finite energy, the cooling of the gas has an easily computable rate, since the average squared velocity per particle decays exponentially, $v_0(t)^2 = \int v^2 P(v,t)\,\mathrm{d}v = v_0(0)^2 \exp(-\lambda t)$ with $\lambda = (1-\alpha^2)/2$.

One of the interesting features of this model, is that it admits a scaling solution of the form $P(v,t) = \frac{1}{v_0(t)} f(v/v_0)$, although the scaling function f cannot be recovered through

a simple analysis of the velocity momenta [81]. In fact, it can be shown that the scaling function that solves the equation, $f(c) = \frac{2}{\pi}\frac{1}{[1+c^2]^2}$, is a distribution whose moments higher than the second diverge. Note that, while the rate of dissipation depends on the restitution coefficient, the scaling solution does not. This is peculiar to the one-dimensional case. In higher dimension [83] the algebraic tails of the solution are still present, but the exponent depends on the restitution coefficient (the tail is narrower and narrower, reducing the inelasticity, and the distribution becomes Gaussian in the elastic limit). The observation of scaling solutions for Maxwell gases is not new. However, for the elastic case, the solutions are not relevant, while in the inelastic case an initial distribution (e.g. an exponential or uniform) rapidly converges to the scaling solution. If we consider an initial distribution with a finite energy per particle and we define $f(c,t) \equiv v_0(t)P(cv_0,t)$, then the equation for the inelastic Maxwell model can be recast into

$$\partial_t f(c,t) + \lambda\partial_c\,(cf(c,t)) = \beta\int \mathrm{d}s\; f(s,t)f(\beta c + (1-\beta)s) - f(c). \qquad (29)$$

This equation can be read as the Boltzmann equation of a new system, which is an inelastic Maxwell gas subjected to a special driving bath (it is often called a 'Gaussian thermostat') given by the term proportional to $\partial_c(cf(c,t))$ [84]. Such a gas performs a stationary dynamics where the energy lost by inelastic collision is compensated by a negative viscosity term, which pushes the particles with a force proportional to their velocity. As we shall see in the following, it is possible to straightforwardly compute the mobility and the diffusion coefficient for a simple stochastic model governed by equation (29). This allows us to explicitly check the validity of the Einstein relation for such a stationary non-equilibrium dynamics.

5.3. Results

Pure systems. In figure 7, left frame, a parametric plot of mobility versus diffusion is displayed for several choices of parameters, showing the linear behaviour analogous to Green–Kubo formulae (the Einstein relation in this case). The same linear behaviour is recovered plotting response versus perturbations in the case of an impulsive shear perturbation (recipe II experiment, see figure 7, right frame). From the slope s of the observed linear behaviour in the response–perturbation graph, one can get the effective temperature $T_{\mathrm{eff}} = 1/(4s)$. We always find $T_{\mathrm{eff}} = T_{\mathrm{g}}$, with $T_{\mathrm{g}} \leqslant T_{\mathrm{b}}$.

Binary mixtures. In the case of binary mixtures, linearity of response–perturbation relations is again verified, in the mobility–diffusion experiment as well as in the current–shear perturbation experiment [30]. Here we review just the former results, i.e. those for the Einstein relation. By successively using as test particle one particle of each species, one obtains the two responses χ_1 and χ_2, and thus the mobilities μ_1 and μ_2. Two independent Einstein relations ($\mu_i = 2D_i/T_i$) are verified, by plotting χ_i versus B_i. In figure 8 we show, as an example, the check of the validity of the Green–Kubo relations using DSMC in spatially homogeneous regime. All the experiments, performed varying the restitution coefficients and the masses of the two components, and with different models and algorithms (homogeneous and inhomogeneous, DSMC and MD), showed identical results, i.e. the linearity of the response–perturbation relation with the effective temperature equal to the granular temperature of the perturbed species.

In figure 9 an even more striking result is portrayed: the mobility–diffusion parametric graph is shown in the case of a single tracer with different properties with respect to a bulk gas ($N_1 = 500$, $N_2 = 1$). In this case the tracer does not significantly perturb the bulk. However, the temperature of the tracer is quite different from the bath temperature as well as from the gas temperature [80]. Again, the effective temperature of the tracer corresponds to

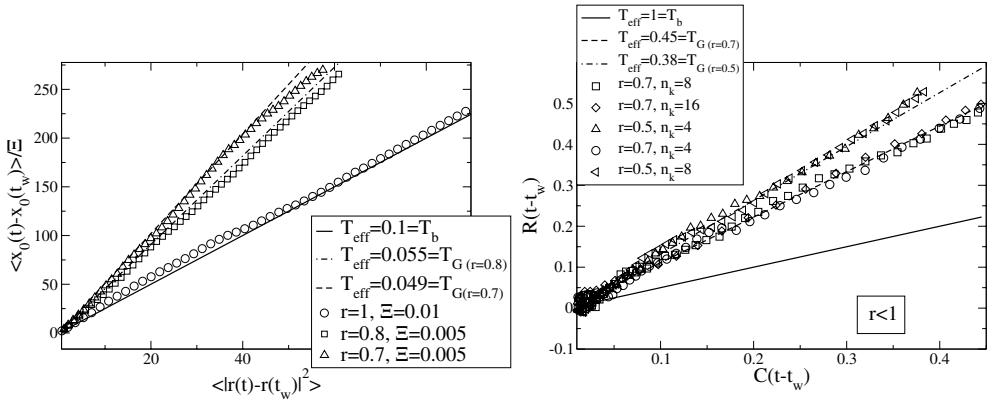

Figure 7. Pure system. Left: parametric plot of $\chi(t, t_w)$ versus $B(t, t_w)$ for the numerical experiment with recipe I (constant force applied on one particle) with $r = 1$, $r = 0.8$ and $r = 0.7$, with heating bath, and for different choices of the intensity Ξ of the perturbation, using $T_b = 0.1$ and $\tau_b = 10$, $N = 500$, $\tau_c = 1$, $t_w = 100$. The results are obtained by averaging over 10 000 realizations. Right: parametric plot of $R(t-t_w)$ versus $C(t-t_w)$ for the numerical experiment with recipe II (impulsive shear perturbation) with $r < 1$, with heating bath, and for different choices of the wavenumber n_k of the perturbation. $T_b = 1$ and $\tau_b = 10$, $N = 500$, $\tau_c = 1$, $\Xi = 0.01$, $n_k = 8$, with averages over 10 000 realizations, using $t_w = 100$.

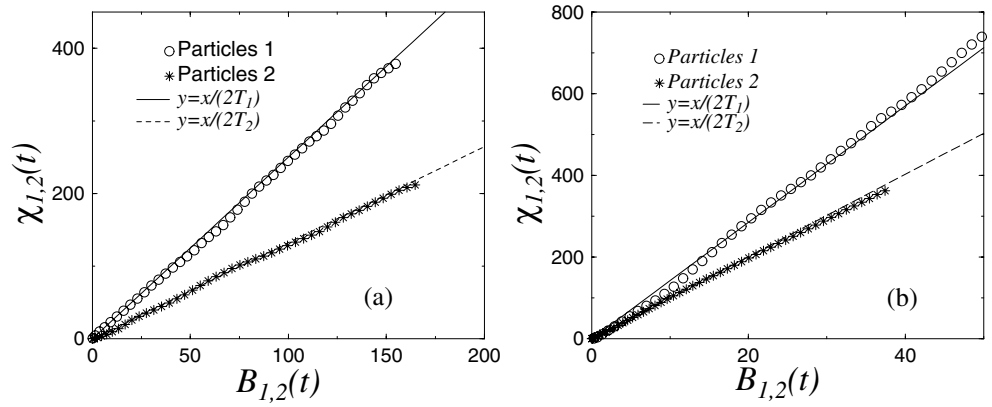

Figure 8. Binary mixture, homogeneous DSMC: mobility versus mean-square displacement; left, $\alpha_{11} = 0.3$, $\alpha_{12} = 0.5$, $\alpha_{22} = 0.7$, $m_2 = 3m_1$, $T_1 \approx 0.2$, $T_2 \approx 0.38$; right, $\alpha_{11} = \alpha_{12} = \alpha_{22} = 0.9$, $m_2 = 5m_1$, $T_1 \approx 0.035$, $T_2 \approx 0.05$. Symbols are numerical data, lines have slope $1/(2T_1)$ and $1/(2T_2)$.

its temperature and not to the temperature of the bath or of the bulk. This is equivalent to saying that a non-perturbing thermometer, used to measure the temperature of a granular gas through fluctuation–response relations, would measure its own temperature and not the bulk temperature.

Exact solution of the inelastic Maxwell model. Here we present a simple calculation on the inelastic one-dimensional Maxwell gas driven by a Gaussian bath where the (non-equilibrium) Einstein relation suggested by numerical simulation holds.

Let us consider a gas of particles performing binary inelastic collision with a constant collision rate (i.e. independent of the precollisional relative velocity) and subjected to a negative

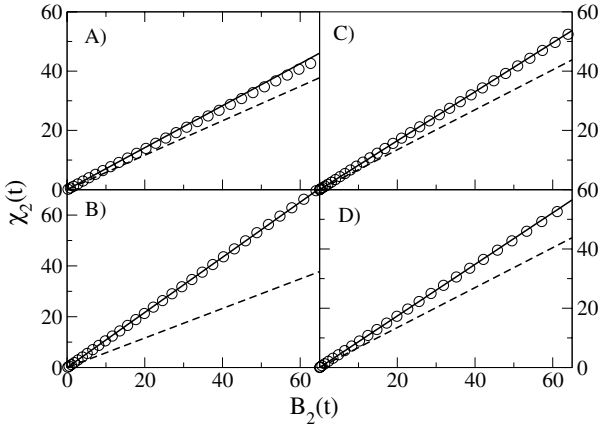

Figure 9. Binary mixture, homogeneous DSMC: mobility versus diffusion of a single particle of mass $m_{\rm tracer}$ in contact with $N = 500$ particles of mass m, immersed in a heat bath (i.e. random kicks plus viscosity). We use the following conventions: $\alpha_{\rm tracer} = \alpha_{12}$ and $\alpha = \alpha_{11}$. Only in case (A) is the tracer also in contact with the external driving heat bath. (A) $m_{\rm tracer} = m$, $\alpha = 0.9$, $\alpha_{\rm tracer} = 0.4$, $T_{\rm g} = 0.86$, $T_{\rm g}^{\rm tracer} = 0.70$; (B) $m_{\rm tracer} = m$, $\alpha = 0.9$, $\alpha_{\rm tracer} = 0.4$, $T_{\rm g} = 0.86$, $T_{\rm g}^{\rm tracer} = 0.46$; (C) $m_{\rm tracer} = 7m$, $\alpha = \alpha_{\rm tracer} = 0.7$, $T_{\rm g} = 0.74$, $T_{\rm g}^{\rm tracer} = 0.60$; (D) $m_{\rm tracer} = 4m$, $\alpha = \alpha_{\rm tracer} = 0.7$, $T_{\rm g} = 0.74$, $T_{\rm g}^{\rm tracer} = 0.57$. The solid line has slope $T_{\rm g}^{\rm tracer}$; the dashed line has slope $T_{\rm g}$.

viscosity force, which drives the gas in a stationary state. The variation of the velocity of a particle after a time Δt is given by

$$v(t + \Delta t) - v(t) = \begin{cases} \lambda v(t)\Delta t & \text{with prob. } 1 - \Delta t \\ -\dfrac{1+\alpha}{2}(v(t) - u) + \lambda v(t)\Delta t & \text{with prob. } \Delta t \end{cases} \tag{30}$$

where u is the velocity of a generic colliding particle, distributed as $f(u)$. The autocorrelation function $A(t_1, t_2) = \overline{v(t_1)v(t_2)}$ (with $t_2 > t_1$) can be computed [85] using (30),

$$\frac{\mathrm{d}}{\mathrm{d}t_2}A(t_1, t_2) = \lim_{\Delta t \to 0} \overline{v(t_1)\frac{v(t_2 + \Delta t) - v(t_2)}{\Delta t}} = \left[\lambda - \frac{1+\alpha}{2}\right] A(t_1, t_2).$$

Since we are considering the stationary case, which is obtained with a negative viscous force exactly balancing the dissipation, i.e. $\lambda = (1 - \alpha^2)/4$, this gives:

$$A(t_1, t_2) = T_{\rm g} \exp\left[-\left(\frac{1+\alpha}{2}\right)^2 (t_2 - t_1)\right].$$

The diffusion coefficient can be computed via the autocorrelation function, obtaining

$$D = \lim_{\tau \to \infty} \int_0^\tau \mathrm{d}\tau'\, A(t_1, t_1 + \tau') = \frac{4T_{\rm g}}{(1+\alpha)^2}. \tag{31}$$

Now, to compute the mobility, we have to apply a small constant force F to a particle, which modifies equation (30) so that $\overline{\frac{\mathrm{d}v(t)}{\mathrm{d}t}} = -\frac{(1+\alpha)^2}{4}\overline{v(t)} + F$. The asymptotic velocity of the tracer yields $4F/(1+\alpha)^2$. This means that the mobility is

$$\mu = \frac{4}{(1+\alpha)^2} = \frac{D}{T_{\rm g}}$$

and the (non-equilibrium) Einstein relation holds.

5.4. Summing up results about granular gases

Approximated analytical results concerning fluctuation–dissipation relations in granular gases have been obtained for the case of mobility and diffusion of a tracer particle in homogeneous cooling granular gases (i.e. without external driving) in [86] and in driven granular gases in [31]. In the cooling case, two main ingredients spoil the usual Green–Kubo formula and lead to a strong reformulation of fluctuation–response relations: they are the strong non-Gaussian behaviour of velocity distribution (in the homogeneous cooling state, HCS, the velocity pdf has exponential tails), and the non-stationarity due to the thermal cooling of the gas. In the HCS, therefore, the Green–Kubo formula must be replaced by a non-linear formula which takes into account these two strong non-equilibrium effects. On the other side, the study of the tracer kinetic equation (Boltzmann–Lorentz equation) for the driven case leads to the conclusion that the only source of deviation from linearity in the fluctuation–response graph may be the velocity non-Gaussian behaviour. However, this is never very pronounced in granular gas driven by a homogeneous source (e.g. when grains are on a table which is vertically vibrated), so that deviations from linearity are negligible. The replacement of T_b by T_g comes quite naturally in the calculations. On the other side, we have shown that in the inelastic Maxwell model, in 1D, where the asymptotic velocity pdf is known to have power law tails, the FD relations are recovered thanks to fortuitous balance of different terms.

The general lesson learnt from simulations and analytical calculation, in the case of stationary dilute granular gas, is that FD relations are difficult to be violated when ergodicity is at work with mostly one characteristic time dominating the dynamics (i.e. the collision time). In such systems, the only cause of (always very small) deviations from usual FD relations is the non-Gaussian behaviour of the velocity statistics, but even in particular cases where velocity pdf is strongly non-Gaussian Green–Kubo formulae may possibly work. The fact that the granular temperature of the measured tracer is the effective temperature is a quite obvious result if the original derivation of Green–Kubo relations is followed, as there the effective temperature appears to be simply $\langle v^2 \rangle$.

6. Conclusions

In this paper we have briefly reviewed the different approaches which have been followed in the last few years in order to define a notion of temperature for granular media. This question is a non-trivial one for such systems where the usual notion of thermodynamic temperature, related to thermal agitation, does not play any obvious role. In these systems the very possibility of consistently constructing a thermodynamics is doubtful due to the fact that energy is lost through internal friction, and gained by non-thermal sources such as tapping or shearing. The dynamical equations, whenever one could be able to write them down explicitly, do not leave the microcanonical or any other known ensemble invariant. Moreover, these systems could never be considered at equilibrium and even the existence of stationary states is not always guaranteed. For instance, often for dense granular media, just as in the case of ageing glasses, a stationary state cannot be reached on experimental timescales.

Despite all these difficulties, in the last few years there have been several contributions which, though not yet completely satisfactory, are interesting because they have opened a new perspective which is worth pursuing in the future.

One of the key concepts has been the notion of dynamic (or effective) temperature, as defined in the framework of the fluctuation–dissipation theorem. Following the remarkable work done on glassy systems where the partial violation of the fluctuation–dissipation theorem has been put in relation with the existence of a so-called dynamical temperature (describing

the slow structural rearrangements of the system), a lot of work has been done along the same lines in the framework of granular media.

Another key contribution is due to Edwards who put forward a very ambitious approach to define a granular 'ensemble' by looking at the so-called blocked (or jammed) configurations.

In this paper we have tried to sum up (in a very partial and maybe subjective way) all these efforts, classifying them with respect to the different regimes a granular medium can be found in: the glassy regime, the liquid-like and the granular gas one.

For the glassy-like regime in particular, we have reviewed Edwards' approach and described a possible path to check its validity for two non-mean-field models: the Kob–Andersen model and the Tetris model. From this and other studies it turns out that the notion of Edwards' compactivity seems to be closely related to that of dynamical temperature: a somewhat surprising but very interesting result, especially because it opens the way to experimental checks of Edwards' hypothesis. For the liquid-like regime we have reported on recent experimental results where an unusual 'thermometer', in the specific case a torsion pendulum, has been used to test the soundness of the temperature concept in a continuously shaken container of tiny beads. Also in this case the temperature has been defined in the framework of the fluctuation–dissipation theorem and its value seems to be consistent with values of the so-called granular temperature, defined in terms of the velocity fluctuations. The relation of this temperature with the one defined in the glassy regime is an open problem. Finally, for the granular gas regime, we have reported on the validity of the fluctuation–dissipation theorem. It turns out that, while inelastic collisions produce large deviations with respect to usual thermodynamics and hydrodynamics, linear response theory and fluctuation–dissipation relations are still valid provided that the 'equilibrium temperature' is replaced by the granular temperature of each component of the gas. There exist deviations with respect to the usual Green–Kubo formula which are due to non-Gaussian behaviour of velocity distribution (in particular for cooling granular gases) and the non-stationarity due to the thermal cooling of the gas. These deviations are indeed very small for driven granular gases.

The picture emerging is still partial, even in each specific regime. It would be important to reinforce in the next years the experimental research in order to check the theoretical predictions and try to bridge some links between the different regimes, even though we expect that the level of universality, for these non-equilibrium systems, is very low. The two extreme regimes seem the most lacking. In particular in the glassy regime it would be important to have some experimental check of Edwards' hypothesis. On the other hand for granular gases there are already many predictions that just call for an experimental check.

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References

[1] Jaeger H M and Nagel S R 1992 *Science* **255** 1523
[2] Jaeger H M, Nagel S R and Behringer R P 1996 *Rev. Mod. Phys.* **68** 1259

[3] For a recent overview see: Coniglio A, Fierro A, Herrmann H J and Nicodemi M (ed) 2003 *Unifying Concepts in Granular Media and Glasses, Proc. of the Homonymous Workshop (Capri, June 2003)* (Amsterdam: Elsevier) at press
[4] Nicodemi M and Coniglio A 1999 *Phys. Rev. Lett.* **82** 916
[5] Barrat A and Loreto V 2000 *J. Phys. A: Math. Gen.* **33** 4401
[6] Josserand C, Tkachenko A V, Mueth D M and Jaeger H M 2000 *Phys. Rev. Lett.* **85** 3632
[7] Barrat A and Loreto V 2001 *Europhys. Lett.* **53** 297
[8] Wang M C and Uhlenbeck G E 1945 *Rev. Mod. Phys.* **17** 323
[9] Kubo R, Toda M and Hashitsume H 1991 *Statistical Physics II: Nonequilibrium Statistical Mechanics* (Berlin: Springer)
[10] Crisanti A and Ritort F 2003 *J. Phys. A: Math. Gen.* **36** R181
[11] Kirkpatrick T R and Thirumalai D 1987 *Phys. Rev.* B **36** 5388
[12] Kirkpatrick T R and Wolynes P G 1987 *Phys. Rev.* A **35** 3072
[13] Cugliandolo L F and Kurchan J 1993 *Phys. Rev. Lett.* **71** 173
Cugliandolo L F and Kurchan J 1994 *J. Phys. A: Math. Gen.* **27** 5749
[14] Cugliandolo L F, Kurchan J and Peliti L 1997 *Phys. Rev.* E **55** 3898
[15] Bouchaud J-P, Cugliandolo L F, Kurchan J and Mézard M 1997 *Spin-Glasses and Random Fields* ed A P Young (Singapore: World Scientific)
[16] Berthier L and Barrat J L 2002 *Phys. Rev. Lett.* **89** 095702
[17] Cugliandolo L 2002 *Dynamics of Glassy Systems* (*Les Houches Lecture Notes*) (*Preprint* cond-mat/0210312)
[18] Liu A J and Nagel S R 1998 *Nature* **396** 21
[19] Barrat A, Kurchan J, Loreto V and Sellitto M 2000 *Phys. Rev. Lett.* **85** 5034
[20] Barrat A, Kurchan J, Loreto V and Sellitto M 2001 *Phys. Rev.* E **63** 051301
[21] Colizza V, Barrat A and Loreto V 2002 *Phys. Rev.* E **65** 050301
[22] Barrat A, Colizza V and Loreto V 2002 *Phys. Rev.* E **66** 011310
[23] Makse H and Kurchan J 2002 *Nature* **415** 614
[24] Brey J J, Prados A and Sanchez-Rey B 2000 *Physica* A **275** 310
[25] Dean D S and Lefèvre A 2001 *J. Phys. A: Math. Gen.* **34** L213
[26] Fierro A, Nicodemi M and Coniglio A 2002 *Europhys. Lett.* **59** 642
Fierro A, Nicodemi M and Coniglio A 2002 *Phys. Rev.* E **66** 061301
[27] Ono I K, O'Hern C S, Langer S A, Liu A J and Nagel S R 2002 *Phys. Rev. Lett.* **89** 095703
[28] Dufty J and Garzò V 2001 *J. Stat. Phys.* **105** 723
[29] Puglisi A, Baldassarri A and Loreto V 2002 *Phys. Rev.* E **66** 061305
[30] Barrat A, Loreto V and Puglisi A 2004 *Physica* A **334** 513
[31] Garzó V 2004 *Physica* A **343** 105
[32] D'Anna G, Mayor P, Barrat A, Loreto V and Nori F 2003 *Nature* **424** 909
[33] Caglioti E, Loreto V, Herrmann H J and Nicodemi M 1997 *Phys. Rev. Lett.* **79** 1575
[34] Caglioti E and Loreto V 1999 *Phys. Rev. Lett.* **83** 4333
[35] For a general introduction see: Jordan D W and Smith P 1977 *Nonlinear Ordinary Differential Equations* (Oxford: Clarendon)
[36] Edwards S F 1994 *Granular Matter: An Interdisciplinary Approach* ed A Mehta (New York: Springer) and references therein
[37] Edwards S F 1991 *Disorder in Condensed Matter Physics* ed J A Blackman and J Taguena (Oxford: Oxford University Press)
[38] Mehta A, Needs R J and Dattagupta S 1992 *J. Stat. Phys.* **68** 1131
[39] Monasson R and Pouliquen O 1997 *Physica* A **236** 395
[40] Monasson R 1995 *Phys. Rev. Lett.* **75** 2847
[41] Kurchan J 1997 *Jamming and Rheology: Constrained Dynamics on Microscopic and Macroscopic Scales* ed S F Edwards, A Liu and S R Nagel (*Preprint* cond-mat/9812347)
http://www.itp.ucsb.edu/online/jamming2/
[42] Nieuwenhuizen Th M 2000 *Phys. Rev.* E **61** 267
[43] Franz S and Virasoro M A 2000 *J. Phys. A: Math. Gen.* **33** 891
[44] Crisanti A and Ritort F 2000 *Europhys. Lett.* **52** 640 (*Preprint* cond-mat/9911226)
[45] Biroli G and Kurchan J 2001 *Phys. Rev.* E **64** 016101
[46] Kurchan J 2000 *J. Phys.: Condens. Matter* **12** 6611
[47] Kob W and Andersen H C 1993 *Phys. Rev.* E **48** 4364
[48] Toninelli C, Biroli G and Fisher D S 2004 *Phys. Rev. Lett.* **92** 185504 (*Preprint* cond-mat/0410647)
[49] Kurchan J, Peliti L and Sellitto M 1997 *Europhys. Lett.* **39** 365

[50] Sellitto M 1998 *Eur. J. Phys.* B **4** 135
[51] Sellitto M and Arenzon J J 2000 *Phys. Rev.* E **62** 7793
[52] De Smedt G, Godrèche C and Luck J M 2002 *Eur. Phys. J.* B **27** 363
[53] Berg J, Franz S and Sellitto M 2002 *Eur. Phys. J.* B **26** 349
[54] Tarjus G and Viot P 2003 *Preprint* cond-mat/0307267
[55] Dean D S and Lefèvre A 2002 *Phys. Rev.* B **65** 220403
[56] Lefèvre A 2002 *J. Phys. A: Math. Gen.* **35** 9037
[57] Dean D S and Lefèvre A 2003 *Phys. Rev. Lett.* **90** 198301
[58] Marty G and Dauchot O 2004 *Preprint* cond-mat/0407017
[59] Caballero G, Lindner A, Ovarlez G, Reydellet G, Lanuza J and Clement E 2004 *Unifying Concepts in Granular Media and Glasses* ed A Coniglio, A Fierro, H J Herrmann and M Nicodemi (*Preprint* cond-mat/0403604)
[60] Langevin P 1908 *C. R. Acad. Sci. Paris* **146** 530
[61] Uhlenbeck G E and Goudsmit S 1929 *Phys. Rev.* **34** 145
[62] Grigera T S and Israeloff N F 1999 *Phys. Rev. Lett.* **83** 5038
[63] Bellon L, Ciliberto S and Laroche C 2001 *Europhys. Lett.* **53** 511
[64] Dixon P K and Durian D J 2003 *Phys. Rev. Lett.* **90** 184302
[65] Yang X, Huan C, Candela D, Mair R W and Walsworth R L 2002 *Phys. Rev. Lett.* **88** 044301
[66] D'Anna G and Gremaud G 2001 *Phys. Rev. Lett.* **87** 254
[67] Pöschel T and Luding S (ed) 2001 *Granular Gases* (*Springer Lecture Notes in Physics* vol 564) (Berlin: Springer)
[68] Pöschel T and Brilliantov N V (ed) 2003 *Granular Gas Dynamics* (*Springer Lecture Notes in Physics* vol 624) (Berlin: Springer)
[69] Williams D R M and MacKintosh F C 1996 *Phys. Rev.* E **54** R9
[70] Puglisi A, Loreto V, Marini Bettolo Marconi U, Petri A and Vulpiani A 1998 *Phys. Rev. Lett.* **81** 3848
Puglisi A, Loreto V, Marini Bettolo Marconi U and Vulpiani A 1999 *Phys. Rev.* E **59** 5582
[71] van Noije T P C and Ernst M H 1998 *Gran. Matter* **1** 57
[72] van Noije T P C, Ernst M H, Trizac E and Pagonabarraga I 1999 *Phys. Rev.* E **59** 4326
Pagonabarraga I, Trizac E, van Noije T P C and Ernst M H 2002 *Phys. Rev.* E **65** 011303
[73] Montanero J M and Santos A 2000 *Gran. Matter* **2** 53
[74] Cafiero R, Luding S and Herrmann H J 2000 *Phys. Rev. Lett.* **84** 6014
[75] Moon S J, Shattuck M D and Swift J B 2001 *Phys. Rev.* E **64** 031303
[76] Cecconi F, Puglisi A and Vulpiani A 2005 *J. Phys.: Condens. Matter* **17** S2715
[77] Bird G A 1994 *Molecular Gas Dynamics and the Direct Simulation of Gas Flows* (Oxford: Clarendon)
[78] Barrat A and Trizac E 2002 *Gran. Matter* **4** 57
[79] Pagnani R, Bettolo Marconi U M and Puglisi A 2002 *Phys. Rev.* E **66** 051304
[80] Martin Ph A and Piasecki J 1999 *Europhys. Lett.* **46** 613
[81] Ben-Naim E and Krapivsky P L 2000 *Phys. Rev.* E **61** R5
[82] Ernst M H 1981 *Phys. Rep.* **78** 1
[83] Krapivsky P L and Ben-Naim E 2002 *J. Phys. A: Math. Gen.* **35** L147
Ernst M H and Brito R 2002 *Europhys. Lett.* **58** 182
[84] Santos A 2003 *Physica* A **321** 442
[85] Ben-Naim E and Krapivsky P 2002 *Phys. Rev.* E **66** 011309
[86] Brey J J, Dufty J W and Ruiz-Montero M M 2003 *Granular Gas Dynamics* ed T Pöschel and N Brilliantov (New York: Springer) p 227 and references therein
Green–Kubo relations for cooling granular gases are also discussed in Brey J J, Ruiz-Montero M J, Maynar P and Garcia de Soria M I 2005 *J. Phys.: Condens. Matter* **17** S2489