## Conformal approach to cylindrical DLA

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#### Abstract

We extend the conformal mapping approach elaborated for the radial diffusion limited aggregation model (DLA) to cylindrical geometry. We introduce in particular a complex function which allows a cylindrical cluster to be grown using as an intermediate step a radial aggregate. The aggregate grown exhibits the same self-affine features as the original cylindrical DLA. The specific choice of the transformation allows us to study the relationship between the radial and the cylindrical geometry. In particular the cylindrical aggregate can be seen as a radial aggregate with particles of size increasing with the radius. On the other hand, the radial aggregate can be seen as a cylindrical aggregate with particles of size decreasing with the height. This framework, which shifts the point of view from the geometry to the size of the particles, can open the way to more quantitative studies on the relationship between radial and cylindrical DLA.


Keywords: irreversible aggregation phenomena (theory), diffusion-limited aggregation (theory), fractal growth (theory)

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## Contents

1. Introduction ..... 2
2. Conformal mapping approach ..... 4
2.1. Radial DLA ..... 4
2.2. Cylindrical DLA ..... 5
3. Construction of the aggregate ..... 10
4. Dimension, self-affinity and self-similarity ..... 13
5. Conclusions ..... 15
Acknowledgment ..... 15
References ..... 15

## 1. Introduction

Since diffusion limited aggregation (DLA) was introduced in 1981 by Witten and Sander [1], an enormous literature has been devoted to it. Its paradigmatic role, concerning a variety of pattern formations in far from equilibrium processes, such as DBM [2], viscous fingering [3], electrodeposition [4], has made DLA one of the models most studied by physicists in the last twenty years. Despite the simplicity of its definition, DLA gives rise to complex branching structures that cannot be described by any small perturbation of a smooth surface.

DLA was first defined on a two-dimensional square lattice. Given a central particle (seed), new particles are added one by one from a far away region. The new particle performs a random walk and, when it touches the aggregate, it becomes part of it. This process is repeated as many times as there are particles composing the cluster.

One can also define the growth process in a cylindrical geometry [5]. The initial seed in this case is a baseline and particles are released from a far away line parallel to the baseline. Since in practice the length of the baseline is finite and one uses periodic boundary conditions, topologically the growth occurs on the surface of a cylinder. We will refer to this model as 'cylindrical DLA'.

The relationship between radial and cylindrical aggregates constitutes a major puzzle for theorists since the fractal and multifractal properties of the aggregate seem definitely to depend (though in a weak way) on the geometry where the growth process occurs. Indeed, whereas the two processes (radial and cylindrical) give rise to basically similar structures, there are small but robust differences that persist to the asymptotic limit [6], and it is not obvious that they are just due to finite size effects. The value of the fractal dimension of radial DLA still represents an open and controversial question [7]-[11], whereas the most accredited one sets approximately around 1.71 . On the other hand, the fractal dimension of the cylindrical DLA seems to be not so sensitive to different measurement techniques and its value, measured by the box-counting method, has been approximated as $1.65[8,13]$. Nevertheless, it is interesting to note that the value of fractal dimension
measured for a circular crosscut in radial aggregates is equal to the one obtained for the intersection set of cylindrical DLA $\left(D_{\mathrm{I}} \sim 0,65\right)[14,15,6]$. Finally, it should be stressed that recent off-lattice simulations using hierarchical map algorithms propose an equivalence between the fractal dimensions of radial and cylindrical DLA [10].

The problem of the correct definition and computation of the fractal dimension is closely linked to the issue of self-similarity. For radial clusters the overall form of aggregates slowly changes during the growth, exhibiting a multi-armed shape and progressively filling the space more uniformly [16]. Indeed the structures generated by simple self-similar scaling models suggest that the lacunarity decreases with increasing size $[17,18,16]$. On the other hand, cylindrical DLA does not present deviations from selfsimilarity; consequently this lacunarity is constant as it grows. Roughly speaking, these discrepancies are probably due to the fact that in cylindrical DLA the cylinder size is fixed and it does not depend on the growth process itself [8], whereas the ratio between cut-offs (size of particles and size of cluster) in radial geometry is constantly changing. In this sense the cylinder geometry offers a conceptual advantage for a theoretical discussion [8, 19], because it defines a unique growth direction and it allows one to vary independently the size of the baseline and the height, which are instead intrinsically linked in the radial geometry.

The two aggregates differ also in the initial non-equilibrium stages of their growth. Although the self-affine scaling regime of the cylindrical DLA constitutes a well-known topic $[12,13,32]$ lying on the surface growth phenomena framework, in the radial geometry the issue of the existence of two different length scales is an open controversial question [20,31].

Recently an elegant representation of radial DLA growth in terms of iterated conformal maps has been introduced by Hastings and Levitov [21,22]. This formulation makes available the powerful tools of analytic function theory, leading to accurate measurements in multifractal properties of the aggregates [23,24] and important theoretical works on the structures of these [26, 25].

The aim of this paper is to extend to the cylindrical geometry the analytic procedure of the conformal mapping elaborated for the radial case. The main idea behind our conformal approach is to map the unitary circle onto the interface of a cylindrical cluster passing through the radial geometry. We introduce, in particular, a complex function that conformally maps the exterior of the unit circle onto the interior of an infinite stripe. Such a function allows us to shift from the radial geometry to the cylindrical one and vice versa. The composition of such a function with a Hastings and Levitov-like function would lead to an analytic map that transforms the exterior of the unit circle onto the complement of a cluster, growing in a stripe with periodic boundary conditions. In this way we can study the growth process of a cylindrical cluster as well as that of its radial deformation: the dimensions are the same for the two aggregates as they are related by isomorphism. The same consideration also applies to the case of a 'real' radial DLA that can be deformed into a cluster growing in a periodic stripe, by composing the functions as before. In this framework the question of the relationship between the radial and the cylindrical DLA appears to be a natural and well-defined problem. In particular the cylindrical aggregate can be seen as a radial aggregate with particles of size increasing with the radius. On the other hand the radial aggregate can be seen as a cylindrical aggregate with particles of size decreasing with the height. This framework, which shifts the point of view from the
geometry to the size of the particles, can open the way to more quantitative studies on the relationship between radial and cylindrical DLA.

The outline of the paper is as follows. In section 2 we recall the conformal mapping model for the radial DLA and we introduce the complex function that transforms the exterior of the unit circle onto an infinite periodic strip. Moreover we present the conformal mapping rules that allow us to build up the cylindrical cluster. In section 3 we focus on the problem of the so-called unphysical particles and we discuss a new procedure for discarding them. In section 4 we will deal with the question of the dimension of cylindrical DLA. We discuss the scaling behaviour of the overall height of a cylindrical cluster as a function of a universal scaling variable. The self-affine and self-similar behaviours of the overall cluster's height are then carefully examined.

## 2. Conformal mapping approach

### 2.1. Radial DLA

We briefly recall the conformal mapping formulation of radial DLA [21].
Let us consider an analytic function that maps the unit circle in the mathematical $w$ plane onto the complement of the cluster of $n-1$ particles in the physical $z$ plane. Such a function is derived from the composition of elementary maps $\varphi_{\lambda, \vartheta}$ :

$$
\begin{equation*}
\Phi^{n}(w)=\Phi^{(n-1)}\left(\varphi_{\lambda_{n}, \vartheta_{n}}(w)\right) \tag{1}
\end{equation*}
$$

The conformal transformation $\varphi_{\lambda, \vartheta}$ maps the exterior of the unit circle to the exterior of the unit circle with a bump of linear scale $\sqrt{\lambda}$ around the point $\mathrm{e}^{\mathrm{i} \vartheta}$. A choice of this function that is free of global distortion is [27]

$$
\begin{equation*}
\varphi_{\lambda, 0}(w)=w^{1 / 2}\left\{\frac{(1+\lambda)}{2 w}(1+w)\left[1+w+w\left(1+\frac{1}{w^{2}}-\frac{2}{w} \frac{1-\lambda}{1+\lambda}\right)^{1 / 2}\right]-1\right\}^{1 / 2} \tag{2}
\end{equation*}
$$

with

$$
\begin{equation*}
\varphi_{\lambda, \vartheta}(w)=\mathrm{e}^{\mathrm{i} \vartheta} \varphi_{\lambda, 0}\left(\mathrm{e}^{-\mathrm{i} \vartheta} w\right) . \tag{3}
\end{equation*}
$$

In order to obtain in the physical $z$ plane particles of fixed size $\sqrt{\lambda_{0}}$ the size of the $n$th bump $\sqrt{\lambda_{n}}$ has to be

$$
\begin{equation*}
\sqrt{\lambda_{n}}=\frac{\sqrt{\lambda_{0}}}{\left|\left(\Phi^{n-1}\right)^{\prime}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|} . \tag{4}
\end{equation*}
$$

We emphasize that this approximation is valid only to the first order. Fluctuation of the magnification factor $\left|\left(\Phi^{n-1}\right)^{\prime}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|$ over the unit circle on the scale of $\sqrt{\lambda_{n}}$ can generate particles of very unequal sizes $[27,28]$ (see also section 3). Furthermore it is immediate that the harmonic probability on the boundary of a real cluster in $z$ translates to an uniform measure on the unit circle:

$$
\begin{equation*}
P(s, \mathrm{~d} s)=\mathrm{d} \vartheta \tag{5}
\end{equation*}
$$

where $z(s)$ is a point on the cluster's interface, and $\mathrm{d} s$ is an infinitesimal arc centred on this point.

In the first papers [21,27] it was also stressed that the scaling of the cluster radius, $R_{n}$, at large $n$ is well characterized by the scaling of the first Laurent coefficient of the mapping function (1). Indeed the Laurent expansion of $\Phi^{n}$ is

$$
\begin{equation*}
\Phi^{n}(w)=F_{1}^{(n)} w+F_{0}^{(n)}+\sum_{k=1}^{+\infty} F_{-k}^{(n)} w^{-k} \tag{6}
\end{equation*}
$$

so that $\Phi^{n}(w) \sim F_{1}^{(n)} w$ as $w \rightarrow \infty$. Since one expects for the radius $R_{n}$ a scaling form $R_{n} \sim(n)^{1 / D} \sqrt{\lambda_{0}}$, where $D$ is the fractal dimension of the cluster, we can assume

$$
\begin{equation*}
F_{1}^{(n)} \sim(n)^{1 / D} \sqrt{\lambda_{0}} . \tag{7}
\end{equation*}
$$

The analytical form of $F_{1}^{(n)}$ is well known [27]:

$$
\begin{equation*}
F_{1}^{(n)}=\prod_{k=1}^{n}\left(1+\lambda_{k}\right)^{1 / 2} \tag{8}
\end{equation*}
$$

The scaling law (7) offers a very convenient way to measure the fractal dimension of the growing cluster [27]; moreover, assuming that the total area of the cluster scales as $A_{n} \sim n \sqrt{\lambda_{0}}[27,28]$, one can write the relation (7) as [25, 28, 29]

$$
\begin{equation*}
R_{n} \equiv F_{1}^{(n)} \sim A_{n}^{1 / D} . \tag{9}
\end{equation*}
$$

Clearly this is true provided that individual particle areas have a sufficiently narrow distribution [28] (see section 3).

### 2.2. Cylindrical DLA

The essential ingredient of our conformal approach for the cylindrical DLA is the modification of the relation (4) which gives rise to a deformed radial cluster with particles of unequal sizes. That cluster will afterwards turn into a cylindrical DLA by means of a suitable function that shifts from the radial geometry to the cylindrical one. In this way we can adapt the conformal theory developed for the radial case to the cylindrical case, without changing the elementary function (2), as instead proposed in [10].

In the radial representation, the conformal mapping (1) can be expressed as

$$
\begin{equation*}
\Phi^{n}(w)=\Phi^{(0)} \circ \varphi_{\lambda_{1}, \vartheta_{1}} \circ \varphi_{\lambda_{2}, \vartheta_{2}} \circ \varphi_{\lambda_{3} \vartheta_{3}} \circ \cdots \circ \varphi_{\lambda_{n}, \vartheta_{n}}(w) . \tag{10}
\end{equation*}
$$

As originally mentioned in [27] the choice of the initial map $\Phi^{(0)}(w)$ is flexible and one can expect the asymptotic shape of the cluster to be independent of this choice. The simplest choice of $\Phi^{(0)}(w)$ was $\Phi^{(0)}(w)=w[21,27]$, which turns the previous definition of the mapping function into

$$
\begin{equation*}
\Phi^{n}(w)=\varphi_{\lambda_{1}, \vartheta_{1}} \circ \varphi_{\lambda_{2}, \vartheta_{2}} \circ \varphi_{\lambda_{3} \vartheta_{3}} \circ \cdots \circ \varphi_{\lambda_{n}, \vartheta_{n}}(w) . \tag{11}
\end{equation*}
$$

Let us now consider the function

$$
\begin{equation*}
\Xi_{0}^{L}(z)=\overline{-\mathrm{i} \frac{L}{2 \pi} \ln (z)}, \tag{12}
\end{equation*}
$$

which maps the exterior of the unit circle in the $z$ plane into the interior of a infinite stripe with periodic boundary conditions in the half-plane $\operatorname{Im}\left(z^{\prime}\right)>0 . L$ is a real parameter


Figure 1. The action of (13) on the complex $z$ plane is schematically represented here. We divide the function into three steps. First box $\left(z \rightarrow z_{a}\right)$ : after a cut along the negative part of the real axis (dashed line), the two parts on the halfplane $\operatorname{Re}(z)<0(\operatorname{Im}(z)>0$ and $\operatorname{Im}(z)<0)$ are rotated in the sense of the arrows, just as in a Chinese fan-like closure. Second box $\left(z_{a} \rightarrow z_{b}\right)$ : the closed Chinese fan is rotated by an angle $=-\pi / 2$. Third box $\left(z_{b} \rightarrow z^{\prime}\right)$ : the fan is rotated with respect the real $z_{b}$ axis. The closure of the Chinese fan is the stripe on the $z^{\prime}$ complex plane where the growth of a cylindrical aggregate occurs.
whose physical meaning is the width of the stripe in the $z^{\prime}$ plane. A similar function was introduced for the first time in [25] with regard to the growth in a channel. If we call the polar coordinates of the $z$ plane $\rho$ and $\theta$ we can rewrite (12) as

$$
\Xi_{0}^{L}(z)=\left\{\begin{array}{l}
x(\rho, \theta)=\frac{L}{2 \pi} \theta  \tag{13}\\
y(\rho, \theta)=\frac{L}{2 \pi} \ln \rho
\end{array}\right.
$$

where $x$ and $y$ are respectively the real and the imaginary parts of $z^{\prime}$. It is possible to imagine the action of (12) on the $z$ plane as the closure of a Chinese fan (see figure 1). For the sake of clarity figure 2 shows a square lattice as the radial plane $z$ is deformed into the plane $z^{\prime}$ by (13).

Our choice of the function (12) seems to be the natural link between radial and cylindrical geometry. Indeed if we compose (12) with (11) we obtain a conformal transformation that maps the unit circle, in the $w$ plane, to the interface of a cylindrical aggregate in the $z^{\prime}$ plane passing through its 'radial version' in the $z$ plane (see figure 3 ):

$$
\begin{equation*}
\Xi_{n}^{L}(w)=\Xi_{0}^{L} \circ \Phi^{n}(w) . \tag{14}
\end{equation*}
$$



Figure 2. Deformation in $z^{\prime}$ of a square lattice in the $z$ complex plane.


Figure 3. Sketch of the conformal approach used to construct a cylindrical aggregate. First we map the unit circle onto the interface of a radial cluster in the $z$ plane, with the usual Hastings and Levitov formula (1). Afterwards, we compose this function with (12). This consists in a mapping from the unit circle to the cluster growing on the $z^{\prime}$ plane.

In fact it is possible to look at the (14) as a particular form of the (10) with the $\Phi^{(0)}$ replaced by $\Xi_{0}^{L}$.

Let us now look, with the help of figures 4 and 5, at the effect of the different transformations introduced. If we grow a cluster according to (1), (4), we have the 'true' radial aggregate in the physical growth space ( $z$; figure $4(\mathrm{a})$ ). On the other hand it is


Figure 4. Radial DLA: (a) radial DLA grown with the rules (1), (4) and the acceptance criterion expressed in section 3. (b) Cylindrical deformation of the same cluster, obtained by composition of (1) with (12); note that in this plane the particle sizes decrease following (15). The simulation was performed with 10000 particles, $\lambda_{0}=0.1$ and cylinder size $L=1$.
possible to see this cluster deformed by (12) in the $z^{\prime}$ plane (figure $4(\mathrm{~b})$ ). Consequently in $z^{\prime}$ the sizes of the particles $\left(\sqrt{\lambda_{0}}\right.$ in $\left.z\right)$ become smaller and smaller as the cluster size increases:

$$
\begin{equation*}
{\sqrt{\lambda_{n}}}^{\left(z^{\prime}\right)}=\frac{\sqrt{\lambda_{0}} L}{2 \pi\left|\Phi^{(n-1)}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}=\frac{\sqrt{\lambda_{0}} L}{2 \pi \mathrm{e}^{(2 \pi / L) \operatorname{Im}\left[\Xi^{(n-1)}\left(\mathrm{e}^{\left.\mathrm{i} \vartheta_{n}\right)}\right)\right]} .} \tag{15}
\end{equation*}
$$

On the other hand, if we want to construct a real cylindrical aggregate in $z^{\prime}$, we have to modify (4) in order to obtain particles of fixed size $\sqrt{\lambda_{0}}$ in this plane. To this end we have to divide the characteristic scale by the Jacobian of the mapping (14):
$\sqrt{\lambda_{n}}=\frac{\sqrt{\lambda_{0}}}{\left|\Xi^{(n-1)^{\prime}}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}=\frac{2 \pi \sqrt{\lambda_{0}}\left|\Phi^{(n-1)}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}{L\left|\Phi^{(n-1)^{\prime}}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}=\frac{2 \pi \sqrt{\lambda_{0}} \mathrm{e}^{(2 \pi / L) \operatorname{Im}\left[\Xi \Xi^{(n-1)}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right]}}{L\left|\Phi^{(n-1)^{\prime}}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}$.


Figure 5. Cylindrical DLA: (a) cylindrical DLA grown with the rules (14), (16) and the acceptance criterion expressed in section 3. (b) Radial aggregate grown with (1), (19), conformal deformation of DLA shown in panel (a) using (12). Here $n=10000, \lambda_{0}=5 \times 10^{-6}$ and $L=1$.

Therefore we can grow a cylindrical cluster in the physical plane $z^{\prime}$ using (14) and (16) (figure 5(a)) just as we grew a radial cluster in the $z$ plane using (1) and (4). In analogy with what was done for the radial case, it is possible to see the cylindrical cluster deformed by the inverse of (12) in the $z$ plane. In this case the size of the particles is increasing exponentially with the height ( $y$ in (13)) of the aggregate (figure 5(b)):

$$
\begin{equation*}
{\sqrt{\lambda_{n}}}^{(z)}=\frac{2 \pi \sqrt{\lambda_{0}}\left|\Phi^{(n-1)}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|}{L}=\frac{2 \pi \sqrt{\lambda_{0}} \mathrm{e}^{(2 \pi / L) \operatorname{Im}\left[\Xi^{(n-1)}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right]}}{L} . \tag{17}
\end{equation*}
$$

The complex potential on the stripe is given by

$$
\begin{equation*}
\Upsilon^{(n)}\left(\overline{z^{\prime}}\right)=\ln \left[\left(\Xi^{(n)}\right)^{-1}\left(\overline{z^{\prime}}\right)\right], \tag{18}
\end{equation*}
$$

so it is quickly verified that $P(s, \mathrm{~d} s)=\mathrm{d} \vartheta$ on the unit circle as in the radial case. Notice that the boundary conditions of the Laplacian field $\nabla P=\Phi^{(n)^{\prime}} /\left|\Phi^{(n)}\right|$ at infinity will be automatically changed from $\nabla P \sim \hat{r} / r$ to $\nabla P \sim \cos t \hat{y}$ [25].

## 3. Construction of the aggregate

On the original work on the radial DLA [21], it was assumed that the rule (4) was sufficient for producing particles with nearly equal areas $\left(\sim \lambda_{0}\right)$. However, as we have noticed in the discussion after (4), this is not true in general: large particles tend to appear within fjords and seal completely large otherwise deeply invaginated regions, where the magnification factor $\Phi^{\prime}$ is not constant around the bump of (3). Since our cylindrical growth model is sensitive to this problem, much more so than the radial one, we introduce here a fast and efficient method for eliminating too large particles.

For radial clusters, a method for the problem of abnormally stretched particles was proposed in [27], by choosing an optimal shape of bump produced by the elementary mapping $\left|\varphi_{\lambda, \vartheta}\right|$; more recently another sophisticated technique was introduced by Stepanov and Levitov [28]. This method consists in evaluating the particle areas with unprecedented accuracy and discarding the particles whose surface exceeds an acceptance threshold. Although the problem of the unphysical particles seems to be crucial in the evaluation of the cluster dimension by means of (9), for radial clusters numerical simulations suggest that the presence of large particles is irrelevant for the cluster size scaling [27, 28].

In the cylindrical case the situation is more subtle. We start by noticing that we can write (16) as

$$
\begin{equation*}
\sqrt{\lambda_{n}}=\frac{{\sqrt{\lambda_{n}}}^{(z)}}{\left|\Phi^{(n-1)^{\prime}}\left(\mathrm{e}^{\mathrm{i} \vartheta_{n}}\right)\right|} \tag{19}
\end{equation*}
$$

The comparison of (19) with (4) clearly shows that it is possible to look at the aggregate grown according to (14) and (16) as a radial cluster with particles whose mean size constantly increases with (17). Since for large $n$ the size of the particles composing the radial version of the cylindrical DLA (see figure 6(a)) is much larger than the one in the original radial model (i.e. ${\sqrt{\lambda_{n}}}^{(z)} \gg \sqrt{\lambda_{0}}$ definitely), the scale $\sqrt{\lambda_{n}}$ in (19) will probably be larger than the one in (4). Consequently we can expect a more important presence of filling particles than in the simple radial model (1), (4). An example of a typical cylindrical cluster (figures 6(a) and (b)) demonstrates that there is a large number of particles that macroscopically affect the growth.

Our method for filtering abnormally large particles is as follows. We recognize that every time we grow a semicircular bump with (3), we generate two new branch cuts in the map $\Phi^{(n)}$. Each branch cut has a pre-image on the unit circle that can be easily derived from (3):

$$
\begin{equation*}
w^{ \pm}=\mathrm{e}^{\mathrm{i} \vartheta^{ \pm}}=\frac{(1-\lambda) \pm 2 \mathrm{i} \sqrt{\lambda}}{(1+\lambda)} \tag{20}
\end{equation*}
$$

These points can be labelled by two indices [30]: $w_{j, n}$. The index $j$ represents the generation when the branch cut was created (i.e. when the $j$ th particle was grown). The index $n$ stands for the generation at which the analysis is being done (i.e. when the cluster has $n$ particles). Indeed, after each iteration, the pre-image of each branch cut moves on the unit circle but its physical positions does not change, so we have a different list of 'exposed' branch cut pre-images $\left\{w_{j, n}\right\}$ each time we grow a particle. Indeed suppose that the list $\left\{w_{j, n-1}\right\}$ is available. In the $n$th generation we grow a new bump around the angle


Figure 6. Cylindrical DLA: (a) cylindrical aggregate obtained by use of (14), (16) and its radial alter ego (b). It is possible to see how the appearance of abnormally stretched particles is frequent in these models. Other parameter simulations have: $n=500, L=10$ and $\lambda_{0}=0.011$.
$\vartheta_{n}$ whose brunch cuts are [27]

$$
\begin{equation*}
\mathrm{e}^{\mathrm{i} \beta^{ \pm}}=\varphi_{\lambda_{n}, \vartheta_{n}}\left(w_{n, n}^{ \pm}\right) . \tag{21}
\end{equation*}
$$

If one or more of the branch cut pre-images in the updated list $\left\{w_{j, n-1}\right\}$ is covered by the $n$th particle (i.e. $w_{j, n-1} \in\left[\beta^{+}, \beta^{-}\right]$for some $j$ ), it will not be registered on the new list $\left\{w_{j, n}\right\}$; otherwise it will be included as

$$
\begin{equation*}
w_{j, n}=\varphi_{\lambda_{n}, \vartheta_{n}}^{-1}\left(w_{j, n-1}\right) . \tag{22}
\end{equation*}
$$

These values, obtained by varying $j$, and the sorted new pair $w_{n, n}^{ \pm}$, will compose the $n$th list. The analytical form of $\varphi_{\lambda, 0}^{-1}$ is [27]

$$
\begin{equation*}
\varphi_{\lambda, 0}^{-1}=\frac{\lambda w^{2} \pm \sqrt{\lambda^{2} w^{4}-w^{2}\left[1-(1+\lambda) w^{2}\right]\left[w^{2}-(1+\lambda)\right]}}{1-(1+\lambda) w^{2}} \tag{23}
\end{equation*}
$$



Figure 7. Controlled growth of the aggregate: (a) covered branch cut distributions (' $P$ (b-c)') for the radial model (1), (4), and (b) for the cylindrical model (14), (16), obtained for different values of the particle size $\sqrt{\lambda_{0}}$; the statistics was over 20 different realizations of the firsts 1000 growth steps setting $L=1$. (c) Scaling behaviour of $F_{1}^{(n)}$ for radial DLA as predicted by (9) for the uncontrolled growth model (solid lines) and for the proposed one (dashed lines), where the axes are set on a logarithmic scale; the expected power-law growth of the radius was drawn for the reader's convenience (dotted curve). (d) Scaling behaviour of the overall height of a cylindrical cluster (see (24) and (25)): it is apparent that the controlled and uncontrolled growth models coincide in the limit $\sqrt{\lambda_{0}} / L \rightarrow 0$; here $L=10$.
and the inverse mapping $\varphi_{\lambda, \vartheta}^{-1}$ is given by $\varphi_{\lambda, \vartheta}^{-1}(w)=\mathrm{e}^{\mathrm{i} \vartheta} \varphi_{\lambda, 0}^{-1}\left(\mathrm{e}^{-\mathrm{i} \vartheta} w\right)$. Notice that the inverse function $\varphi_{\lambda, \vartheta}^{-1}$ is analytic on the unit circle only outside the $\operatorname{arc}\left[\beta^{+}, \beta^{-}\right]$.

The branch cut covered distributions for radial model (1), (4) and cylindrical model (14), (16) are displayed in panels (a) and (b) of figure 7 respectively. The two distributions are equally peaked about one covered branch cut and seem to show similar shapes and the same power-law decreasing trend on the tails. Furthermore it is apparent that such tails tend to become less pronounced as the particle size decreases and to completely disappear in the limit $\sqrt{\lambda_{0}} \rightarrow 0, \sqrt{\lambda_{0}} / L \rightarrow 0$ (not shown).

In order to solve the problem of unphysical particles, we truncate the tails of the distributions $P(\mathrm{~b}-\mathrm{c})$ for both radial and cylindrical models, discarding the particles covering more than three branch cuts. This operation is relatively fast because we know at each generation the list of the exposed branch cut pre-images on the unit circle (see (22)). Then, for each particle added, we calculate how many branch cuts of the previous generation the new particle covers. If this number is $\leq 3$ the particle is accepted; otherwise it is discarded and a new attempt at particle growth is made.

As noticed in [28], it is not clear a priori whether the growth which discards too large particles is macroscopically equivalent to that without restrictions and whether such
models reproduce the same lattice aggregates. However for radial and cylindrical cases the only significant proof is the comparison between the scaling exponents derived numerically and the dimensions accredited.

For this reason we plot the quantities $F_{1}^{(n)}$ and $\ln F_{1}^{(n)}$ in panels (c) and (d) of figure 7, related respectively to the overall radius of a radial cluster (9) and to the height of a cylindrical DLA (see (25) in section 4). Solid lines refer to models on which any particle selection criterion is absent, whereas the dashed ones are related to the proposed controlled growth procedure. We first note that the slopes of both the curves remain unchanged by the presence or the absence of too large particles: this leads us to conclude that the method of discarding particles covering more than three branch cuts does not compromise the data extrapolation of the dimension. Moreover these are in accordance with the accredited values present in the literature: 1.71 for radial DLA and 1.67 for cylindrical aggregates (see the next section). However we stress that our method allows us to filter out the fluctuations due to the presence of filling particles that inevitably affect the curves in panels (c) and (d) of figure 7; such noisy behaviour becomes more and more apparent in the cylindrical case as the ratio $\sqrt{\lambda_{0}} / L$ tends to 1 .

## 4. Dimension, self-affinity and self-similarity

As already noticed, the first Laurent coefficient scales as the radius $R_{n}$ of the aggregate $[21,27]$ and it consequently provides a useful tool for studying the scaling behaviour of the cluster in the radial case. This property also applies to the cylindrical cluster in its radial deformation, so that the overall size of the cluster grown in the $z$ plane (figure $5(\mathrm{~b})$ ) is well characterized by $F_{1}^{(n)}$. Moreover the radius in the $z$ plane is related to the height $(y)$ in the physical plane $z^{\prime}$ by the second of (13), so that the logarithm of $F_{1}^{(n)}$ should represent the overall height of the cluster. Under the hypothesis that the cylindrical DLA is self-similar in squares of size $L / \sqrt{\lambda_{0}}$, the mean height of a cylindrical cluster composed of $n$ particles growing in a stripe of size $L / \sqrt{\lambda_{0}}$ scales as

$$
\begin{equation*}
y(n) \sim \frac{{\sqrt{\lambda_{0}}}^{D}}{L^{D-1}} n \tag{24}
\end{equation*}
$$

where $\sqrt{\lambda_{0}}$ is the linear size of the particles. We can thus derive the scaling relation for the first Laurent coefficient in the case of a cylindrical cluster:

$$
\begin{equation*}
\ln F_{1}^{(n)} \sim 2 \pi\left(\frac{\sqrt{\lambda_{0}}}{L}\right)^{D} n \tag{25}
\end{equation*}
$$

It is important to remark that the self-similar growth (24) of a cylindrical DLA is attained only in the steady state regime [13], i.e. for $n \gg\left(L / \sqrt{\lambda_{0}}\right)^{D}$.

Along the same lines as were followed in $[26,31]$ for radial clusters, we can argue that $\ln F_{1}^{(n)}\left(\sqrt{\lambda_{0}} / L\right)$ converges to a fixed point function $\left(\ln F_{1}\right)^{*}$ of the single scaling variable $x=\left(\sqrt{\lambda_{0}} / L\right)^{D} n$, attaining the linear regime (24) asymptotically $(x \gg 1)$. In figure 8 we present the average height $y(n)=(L / 2 \pi) \ln F_{1}^{(n)}\left(\sqrt{\lambda_{0}} / L\right)$ as a function of $x$ for a typical DLA realization for $L=1$ and $\sqrt{\lambda_{0}}$ ranging from $10^{-3}$ to $10^{-7}$. Two different collapses are presented, setting $D=1.67$ and 1.71.


Figure 8. Overall height of a cylindrical DLA obtained for different values of the particle size $\sqrt{\lambda_{0}}$; the collapse onto the universal scaling function (26) is apparent under the rescaling $x=n\left(\sqrt{\lambda_{0}} / L\right)^{D}$; the two different behaviours of $g(x)$ in (26) are also displayed for comparison: $\sim x^{1.35 \pm 0.03}$ (dotted line), and $\sim x$ (dashed line). The statistics was over 20 different realizations with $L=1$. Two different collapses are displayed corresponding to $D=1.67$ and $D=1.71$ (shifted for clarity by one decade to the right).

It is evident how all the curves reasonably collapse onto a unique scaling function

$$
\begin{equation*}
y(x)=L g(x) \tag{26}
\end{equation*}
$$

with

$$
g(x) \sim \begin{cases}x^{1+\beta} & y \ll 1  \tag{27}\\ x & y \gg 1\end{cases}
$$

and $\beta=0.35 \pm 0.03$, obtained by averaging over the slopes of different curves.
Several remarks are in order:
(i) the convergence of $\ln F_{1}^{(n)}$ to the fixed point function is obtained infinitesimally close to the cylinder baseline, for which $\ln F_{1}^{(n)}=0$;
(ii) the fixed point function exists already for $x \ll 1$ : in this transient regime the height of the aggregate exhibits the power-law behaviour $\sim x^{\alpha}$, with $\alpha \simeq \frac{4}{3}$;
(iii) the predicted linear behaviour is reached for $x \geq 1$.

The first two properties are also satisfied by radial clusters [31], while the third is a specific feature of cylindrical aggregates.

The property (i) refers to the stage of growth before the collapse and can be roughly explained as follows.

For cylindrical self-similar clusters, if we define as $n_{\mathrm{c}}$ the number of particles required to obtain one-layer coverage of the original circular interface, then $n_{\mathrm{c}} \sim\left(\sqrt{\lambda_{0}} / L\right)^{(1-D)}$. Therefore $n_{\mathrm{c}}\left(\sqrt{\lambda_{0}} / L\right)^{D} \rightarrow 0$ if $\sqrt{\lambda_{0}} / L \rightarrow 0$.

The properties (ii) and (iii) refer to the fixed point function: in particular (ii) refers to the self-affine initial growth regime of the aggregate. The issue of self-affine growth of cylindrical DLA has been the subject of previous theoretical [32] and numerical investigations $[12,13]$, both showing that the average cluster height grows as

$$
\begin{equation*}
y(n) \sim L^{\nu_{\|} / \nu_{\perp}} f\left(n\left(\frac{\sqrt{\lambda_{0}}}{L}\right)^{1 / \nu_{\perp}}\right), \tag{28}
\end{equation*}
$$

where $\nu_{\|}$is the scaling exponent of a single DLA tree in the transverse direction, $\nu_{\perp}$ is the scaling exponent in the growth direction and $f(x)$ is a scaling function that behaves like $\sim x^{\nu_{\|} /\left(1-\nu_{\perp}\right)}$ for $x \ll 1[33]$ and attains the linear regime for $x \gg 1$.

Our results indicate that the DLA dimension appears as a scaling exponent even before the self-similar regime of growth (24) sets in, thus replacing both $\nu_{\|}$and $\nu_{\perp}$ in (28). Note that setting $\nu_{\|}=\frac{2}{3}$ and $\nu_{\perp}=\frac{1}{2}$ in accordance with [12], $g(x)$ coincides with $f(x)$ in (28).

A last remark concerns the estimate of the fractal dimension of the cylindrical aggregate, as obtained with our conformal approach. From figure 8 it is evident how the two data collapses are not sufficiently sharp for discriminating between $D=1.67$ and 1.71. Additional numerical work (especially for large sizes, i.e. very small values of $\sqrt{\lambda_{0}}$ ) would be necessary to check whether the asymptotic fractal dimension of the cylindrical DLA could be $D=1.71$, as in the radial case, as suggested in [10].

## 5. Conclusions

In this paper we examined carefully the numerical procedure for generating the conformal maps in the cylindrical case. We have defined the conformal map as the composition of two maps: one that maps the cylindrical aggregate in a radial one, and another that maps the radial aggregate in the circle. This procedure opens up new prospects for the comprehension of the relationship between the aggregates growing on different geometries. This also offers a conceptual advantage on the definition of the 'right way' to calculate the fractal dimension of the DLA.

Moreover we have proposed an auto-affine scaling relation for the earlier stages of growth of the aggregate, in which the dimension of the aggregate appears as a scaling exponent.

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## References

[1] Witten T A and Sander H J, 1981 Phys. Rev. Lett. 471400
[2] Niemeyer L, Pietronero L and Wiessmann H J, 1984 Phys. Rev. Lett. 521033
[3] Paterson L, 1984 Phys. Rev. Lett. 521621
[4] Brady R M and Ball R C, 1984 Nature 309225 Grier D, Ben-Jacob E, Clarke R and Sander L M, 1986 Phys. Rev. Lett. 561264
[5] Meakin P, 1983 Phys. Rev. A 272616
[6] Mandelbrot B B, Vespignani A and Kaufman H, 1995 Europhys. Lett. 32199
[7] Meakin P and Tolman S, 1989 Fractal's Origin and Properties ed L Pietronero (New York: Plenum)
[8] Erzan A, Pietronero L and Vespignani A, 1986 Rev. Mod. Phys. 67545
[9] Mandelbrot B B, Kol B and Aharony A, 2002 Phys. Rev. Lett. 88055501
[10] Ball R C, Bowler N E, Sander L M and Somfai E, 2002 Phys. Rev. E 66026109
[11] For a recent paper with a good discussion of the existing literature see: Menshutin A Y and Schur L N, 2005 Preprint cond-mat/0504338
[12] Meakin P and Family F, 1986 Phys. Rev. A 342558
[13] Evertsz C, 1990 Phys. Rev. A 411830
[14] Kolb M, 1985 J. Physique 46 L631
[15] Arneodo A, Argoul F, Bacry E, Muzy J F and Tabard M, 1992 Phys. Rev. Lett. 683456
[16] Mandelbrot B B, 1992 Physica A 19195
[17] Ossadnik P, 1991 Physica A 176454
[18] Meakin P and Havlin S, 1992 Phys. Rev. A 19195
[19] Hastings M B, 2001 Phys. Rev. E 64046104
[20] Plischke M and Rácz Z, 1981 Phys. Rev. Lett. 471400
Plischke M and Rácz Z, 1983 Phys. Rev. B 275686
Meakin P and Sander L M, 1984 Phys. Rev. Lett. 542054
Meakin P, 1985 Phys. Rev. A 333371
[21] Hastings M B and Levitov L S, 1996 Physica D 116244
[22] Hastings M B, 1997 Phys. Rev. E 55136
[23] Jensen M H, Levermann A, Mathiensen J and Procaccia I, 2002 Phys. Rev. E 65046109
[24] Jensen M H, Mathiensen J and Procaccia I, 2002 Phys. Rev. E 67042402
[25] Barra F, Davidovitch B and Procaccia I, 2002 Phys. Rev. E 65046144
[26] Davidovitch B, Levermann A and Procaccia I, 2000 Phys. Rev. E 62 R5919
[27] Davidovitch B, Hentschel H G, Olami Z, Procaccia I, Sander L M and Somfai E, 1999 Phys. Rev. E 591368
[28] Stepanov M G and Levitov L S, 2001 Phys. Rev. E 63061102
[29] Hentschel H G, Levermann A and Procaccia I, 2002 Phys. Rev. E 66016308
[30] Jensen M H, Levermann A, Mathiensen J and Procaccia I, 2002 Phys. Rev. E 65046109
[31] Davidovitch B and Procaccia I, 2000 Phys. Rev. Lett. 853608
[32] Nadal J P, Derrida B and Vannimenus J, 1982 J. Physique 431561
[33] Rácz P and Vicseck T, 1983 Phys. Rev. Lett. 512382

